

A duality map for the quantum symplectic double

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Abstract

This paper is a continuation of the author's work with Kim [3], which provided a natural q -deformation of Fock and Goncharov's canonical basis for the coordinate ring of a cluster variety associated to a punctured surface. Here we consider a cluster variety called the symplectic double, defined for an oriented disk with finitely many marked points on its boundary. We construct a natural map from the tropical integral points of the symplectic double into its quantized algebra of rational functions. Using this construction, we extend the results of [3] to the case of a disk with marked points.

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1 Introduction

1.1 The cluster Poisson variety

Cluster varieties are geometric objects introduced and studied by Fock and Goncharov in a series of recent papers [6, 7, 8, 9]. They arise naturally as moduli spaces of geometric structures on a compact oriented surface with finitely many marked points on its boundary. In the original paper on the subject, Fock and Goncharov defined a version of the moduli space of $PGL_2(\mathbb{C})$ -local systems on such a surface S [6]. This space is denoted $\mathcal{X}_{PGL_2,S}(\mathbb{C})$. It is naturally identified with a particular kind of cluster variety called a *cluster Poisson variety* [6].

One of the main results from the work of Fock and Goncharov states that the algebra of regular functions on the space $\mathcal{X}_{PGL_2,S}(\mathbb{C})$ possesses a canonical basis with many special properties. To construct this canonical basis, Fock and Goncharov considered another space, denoted $\mathcal{A}_{SL_2,S}(\mathbb{Z}^t)$. This object can be understood as a tropicalization of a certain moduli space of $SL_2(\mathbb{C})$ -local systems associated to the surface S . In the case where S is a surface without marked points, Fock and Goncharov constructed a canonical map

$$\mathbb{I}_A : \mathcal{A}_{SL_2,S}(\mathbb{Z}^t) \rightarrow \mathcal{O}(\mathcal{X}_{PGL_2,S})$$

from this space into the algebra of regular functions on $\mathcal{X}_{PGL_2,S}(\mathbb{C})$, and they showed that the image of this map is a vector space basis for the algebra of regular functions.

This canonical basis construction is expected to be closely related to Lusztig's canonical bases in the theory quantum groups [16], as well as the work of Gross, Hacking, Keel, and Kontsevich on canonical bases for cluster algebras [14]. It has also found applications in mathematical physics where it was used by Gaiotto, Moore, and Neitzke to calculate BPS degeneracies in certain four-dimensional supersymmetric quantum field theories [13].

An important feature of the cluster Poisson variety is that this object carries a canonical Poisson structure and can be canonically quantized [8, 9]. In other words, there is a family of

noncommutative algebras $\mathcal{O}_q(\mathcal{X}_{PGL_2,S})$ depending on a parameter q such that $\mathcal{O}_q(\mathcal{X}_{PGL_2,S})$ coincides with the commutative algebra $\mathcal{O}(\mathcal{X}_{PGL_2,S})$ when $q = 1$. In recent joint work with Kim [3], the author showed that Fock and Goncharov's canonical basis construction extends naturally to the quantum setting.

Theorem 1.1 ([3]). There exists a natural map

$$\mathbb{I}_{\mathcal{A}}^q : \mathcal{A}_{SL_2,S}(\mathbb{Z}^t) \rightarrow \mathcal{O}_q(\mathcal{X}_{PGL_2,S})$$

which reduces to $\mathbb{I}_{\mathcal{A}}$ in the classical limit $q = 1$.

In fact, the main result of [3] was a much more detailed statement which established a number of properties of $\mathbb{I}_{\mathcal{A}}^q$ conjectured in [6, 8]. The map $\mathbb{I}_{\mathcal{A}}^q$ is expected to be important in physics where it is related to the “protected spin character” introduced by Gaiotto, Moore, and Neitzke [13].

1.2 The symplectic double

The main goal of the present paper is to extend the results described above to another cluster variety called the *cluster symplectic variety* or *symplectic double*. This cluster variety was introduced by Fock and Goncharov and plays a key role in their work on quantization of cluster varieties [9]. As before, one can associate, to a compact oriented surface S with finitely many marked points on its boundary, a certain moduli space $\mathcal{D}_{PGL_2,S}(\mathbb{C})$ of local systems, and this moduli space is naturally identified with the symplectic double [10, 2].

The main result of [1] was the construction of a dual space $\mathcal{D}_{PGL_2,S}(\mathbb{Z}^t)$. This space can be understood as a tropicalization of $\mathcal{D}_{PGL_2,S}(\mathbb{C})$. Just as the tropical space $\mathcal{A}_{SL_2,S}(\mathbb{Z}^t)$ parametrizes a canonical basis for the algebra of regular functions on $\mathcal{X}_{PGL_2,S}(\mathbb{C})$, the space $\mathcal{D}_{PGL_2,S}(\mathbb{Z}^t)$ parametrizes a canonical collection of functions on $\mathcal{D}_{PGL_2,S}(\mathbb{C})$. More precisely, there is a canonical map

$$\mathbb{I}_{\mathcal{D}} : \mathcal{D}_{PGL_2,S}(\mathbb{Z}^t) \rightarrow \mathbb{Q}(\mathcal{D}_{PGL_2,S})$$

from this set into the field of rational functions on $\mathcal{D}_{PGL_2,S}(\mathbb{C})$, and there is an explicit formula expressing this map in terms special polynomials called *F-polynomials* from the work of Fomin and Zelevinsky [12].

In general, the function obtained by applying the map $\mathbb{I}_{\mathcal{D}}$ to a point of $\mathcal{D}_{PGL_2,S}(\mathbb{Z}^t)$ is not regular, and so this map does not provide a canonical basis for the coordinate ring $\mathcal{O}(\mathcal{D}_{PGL_2,S})$. However, the relation to *F*-polynomials suggests that this construction may have applications in the theory of cluster algebras. Understanding the relationship between the map $\mathbb{I}_{\mathcal{D}}$, canonical bases for cluster algebras, and the physical ideas of Gaiotto, Moore, and Neitzke [13] is an interesting problem for future research.

Like the cluster Poisson variety, the symplectic double admits a natural deformation quantization [9]. In other words, there is a family of noncommutative algebras $\mathcal{D}_{PGL_2,S}^q$ depending on a parameter q such that we recover the function field $\mathbb{Q}(\mathcal{D}_{PGL_2,S})$ when $q = 1$. The main result of the present paper is a canonical q -deformation of the map $\mathbb{I}_{\mathcal{D}}$ in the important special case where S is a disk with finitely many marked points on its boundary.

Theorem 1.2. When S is a disk with finitely many marked points, there exists a natural map

$$\mathbb{I}_{\mathcal{D}}^q : \mathcal{D}_{PGL_2, S}(\mathbb{Z}^t) \rightarrow \mathcal{D}_{PGL_2, S}^q$$

which reduces to $\mathbb{I}_{\mathcal{D}}$ in the classical limit $q = 1$.

The existence of the map $\mathbb{I}_{\mathcal{D}}^q$ was essentially conjectured by Fock and Goncharov in Conjecture 3.6 of [9]. However, this conjecture was formulated before it was understood that the classical map $\mathbb{I}_{\mathcal{D}}$ naturally maps into the field of rational functions. In light of [1], we know $\mathbb{I}_{\mathcal{D}}^q$ should map into the quantized algebra of *rational* functions, rather than the quantized algebra of regular functions as predicted in [9].

1.3 Extension of previous results

The cluster symplectic variety is a symplectic space which contains the cluster Poisson variety as a Lagrangian subspace [9]. The embedding of the cluster Poisson variety into the cluster symplectic variety induces a map $\pi^* : \mathbb{Q}(\mathcal{D}_{PGL_2, S}) \rightarrow \mathbb{Q}(\mathcal{X}_{PGL_2, S})$ of function fields. In fact, for each q , we have a map $\pi^q : \mathcal{D}_{PGL_2, S}^q \rightarrow \mathcal{X}_{PGL_2, S}^q$ where $\mathcal{X}_{PGL_2, S}^q$ denotes the quantized algebra of rational functions on the cluster Poisson variety. One also has a natural embedding $\varphi : \mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t) \hookrightarrow \mathcal{D}_{PGL_2, S}(\mathbb{Z}^t)$ where $\mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t)$ is a version of the tropical space $\mathcal{A}_{SL_2, S}(\mathbb{Z}^t)$.

By restricting $\mathbb{I}_{\mathcal{D}}^q$ to the subspace $\mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t)$ of $\mathcal{D}_{PGL_2, S}(\mathbb{Z}^t)$, we obtain a map $\mathbb{I}_{\mathcal{A}}^q$, which fits into the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t) & \xrightarrow{\mathbb{I}_{\mathcal{A}}^q} & \mathcal{X}_{PGL_2, S}^q \\ \varphi \downarrow & & \uparrow \pi^q \\ \mathcal{D}_{PGL_2, S}(\mathbb{Z}^t) & \xrightarrow{\mathbb{I}_{\mathcal{D}}^q} & \mathcal{D}_{PGL_2, S}^q \end{array}$$

In fact, we will see in Theorem 4.7 that this map $\mathbb{I}_{\mathcal{A}}^q$ satisfies several properties conjectured in [6, 8]. Thus we have an extension of the construction of [3], which was valid only when S is a surface *without marked points*, to the case where S is a disk with finitely many marked points.

Interestingly, our construction of the map $\mathbb{I}_{\mathcal{A}}^q$ in the disk case uses several tools that were not present in [3]. These include the theory of quantum cluster algebras [4], the notion of quantum F -polynomial from [20], and the work of Muller on skein algebras [17]. The original construction of $\mathbb{I}_{\mathcal{A}}^q$ in [3] was instead based on the “quantum trace map” introduced by Bonahon and Wong [5]. Recent results of Lê [15] suggest that these two approaches are in fact equivalent.

1.4 Organization

The rest of this paper is organized as follows. In Section 2, we define the cluster Poisson variety, the cluster symplectic variety, and their q -deformations using the quantum dilogarithm

function. In Section 3, we define the skein algebra and review the results of Muller [17], which relate skein algebras to quantum cluster algebras. In Section 4, we construct the map $\mathbb{I}_{\mathcal{A}}^q$ and prove a number of conjectured properties of this map. In Section 5, we define the map $\mathbb{I}_{\mathcal{D}}^q$ and discuss its relation to $\mathbb{I}_{\mathcal{A}}^q$.

2 Quantum cluster varieties

2.1 Seeds and mutations

Cluster varieties are geometric objects defined using combinatorial ideas from Fomin and Zelevinsky's theory of cluster algebras [11]. One of the main concepts needed to define cluster varieties is the notion of a seed.

Definition 2.1. A *seed* $\mathbf{i} = (\Lambda, \{e_i\}_{i \in I}, \{e_j\}_{j \in J}, (\cdot, \cdot))$ is a quadruple where

1. Λ is a lattice with basis $\{e_i\}_{i \in I}$.
2. $\{e_j\}_{j \in J}$ is a subset of the basis.
3. (\cdot, \cdot) is a \mathbb{Z} -valued skew-symmetric bilinear form on Λ .

A basis vector e_j with $j \in J$ is said to be *mutable*, while a basis vector e_i with $i \in I - J$ is said to be *frozen*. Note that if we are given a seed, we can form a skew-symmetric integer matrix with entries $\varepsilon_{ij} = (e_i, e_j)$ ($i, j \in I$).

The second main concept that we need to define cluster varieties is the notion of mutation. For any integer n , let us write $[n]_+ = \max(0, n)$.

Definition 2.2. Let $\mathbf{i} = (\Lambda, \{e_i\}_{i \in I}, \{e_j\}_{j \in J}, (\cdot, \cdot))$ be a seed and e_k ($k \in J$) a mutable basis vector. Then we define a new seed $\mathbf{i}' = (\Lambda', \{e'_i\}_{i \in I}, \{e'_j\}_{j \in J}, (\cdot, \cdot)')$ called the seed obtained by *mutation* in the direction of e_k . It is given by $\Lambda' = \Lambda$, $(\cdot, \cdot)' = (\cdot, \cdot)$, and

$$e'_i = \begin{cases} -e_k & \text{if } i = k \\ e_i + [\varepsilon_{ik}]_+ e_k & \text{if } i \neq k. \end{cases}$$

It is straightforward to calculate the change of the matrix ε_{ij} under a mutation of seeds.

Proposition 2.3. A mutation in the direction k changes the matrix ε_{ij} to the matrix

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } k \in \{i, j\} \\ \varepsilon_{ij} + \frac{|\varepsilon_{ik}| \varepsilon_{kj} + \varepsilon_{ik} |\varepsilon_{kj}|}{2} & \text{if } k \notin \{i, j\}. \end{cases}$$

The operation of mutation is involutive. That is, if we mutate the seed \mathbf{i}' in the direction of e'_k , we recover the original seed \mathbf{i} . Two seeds will be called *mutation equivalent* if they are related by a sequence of mutations. We will denote the mutation equivalence class of a seed \mathbf{i} by $[\mathbf{i}]$.

2.2 The quantum dilogarithm and quantum tori

To define quantum cluster varieties, we employ the following special function.

Definition 2.4. The *quantum dilogarithm* is the formal power series

$$\Psi^q(x) = \prod_{k=1}^{\infty} (1 + q^{2k-1}x)^{-1}.$$

We will study the quantum dilogarithm as a function on the following algebra of q -commuting variables.

Definition 2.5. Let Λ be a lattice equipped with a \mathbb{Z} -valued skew-symmetric bilinear form (\cdot, \cdot) . Then the *quantum torus algebra* is the noncommutative algebra over $\mathbb{Z}[q, q^{-1}]$ generated by variables Y_v ($v \in \Lambda$) subject to the relations

$$q^{-(v_1, v_2)} Y_{v_1} Y_{v_2} = Y_{v_1 + v_2}.$$

This definition allows us to associate to any seed $\mathbf{i} = (\Lambda, \{e_i\}_{i \in I}, \{e_j\}_{j \in J}, (\cdot, \cdot))$, a quantum torus algebra $\mathcal{X}_{\mathbf{i}}^q$. The set $\{e_j\}_{j \in J}$ provides a set of generators $X_j^{\pm 1}$ given by $X_j = Y_{e_j}$ for this algebra. They obey the commutation relations

$$X_i X_j = q^{2\varepsilon_{ij}} X_j X_i.$$

This algebra $\mathcal{X}_{\mathbf{i}}^q$ satisfies the Ore condition from ring theory, so we can form its noncommutative fraction field $\widehat{\mathcal{X}}_{\mathbf{i}}^q$. In addition to associating a quantum torus algebra to every seed, we use the quantum dilogarithm to construct a natural map $\widehat{\mathcal{X}}_{\mathbf{i}'}^q \rightarrow \widehat{\mathcal{X}}_{\mathbf{i}}^q$ whenever two seeds \mathbf{i} and \mathbf{i}' are related by a mutation.

Definition 2.6.

1. The automorphism $\mu_k^{\#} : \widehat{\mathcal{X}}_{\mathbf{i}}^q \rightarrow \widehat{\mathcal{X}}_{\mathbf{i}}^q$ is given by conjugation with $\Psi^q(X_k)$:

$$\mu_k^{\#} = \text{Ad}_{\Psi^q(X_k)}.$$

2. The isomorphism $\mu'_k : \widehat{\mathcal{X}}_{\mathbf{i}'}^q \rightarrow \widehat{\mathcal{X}}_{\mathbf{i}}^q$ is induced by the natural lattice map $\Lambda' \rightarrow \Lambda$.
3. The mutation map $\mu_k^q : \widehat{\mathcal{X}}_{\mathbf{i}'}^q \rightarrow \widehat{\mathcal{X}}_{\mathbf{i}}^q$ is the composition $\mu_k^q = \mu_k^{\#} \circ \mu'_k$.

Note that conjugation by $\Psi^q(X_k)$ produces a priori a formal power series. We will see below that this construction in fact provides a map $\widehat{\mathcal{X}}_{\mathbf{i}'}^q \rightarrow \widehat{\mathcal{X}}_{\mathbf{i}}^q$ of skew fields.

2.3 The quantum double construction

As explained in [9], it is natural to embed the above construction in a larger one. If $\mathbf{i} = (\Lambda, \{e_i\}_{i \in I}, \{e_j\}_{j \in J}, (\cdot, \cdot))$ is any seed, then we can form the “double” $\Lambda_{\mathcal{D}} = \Lambda_{\mathcal{D}, \mathbf{i}}$ of the lattice Λ given by the formula

$$\Lambda_{\mathcal{D}} = \Lambda \oplus \Lambda^{\vee}$$

where $\Lambda^{\vee} = \text{Hom}(\Lambda, \mathbb{Z})$. The basis $\{e_i\}$ for Λ provides a dual basis $\{f_i\}$ for Λ^{\vee} , and hence we have a basis $\{e_i, f_i\}$ for $\Lambda_{\mathcal{D}}$. Moreover, there is a natural skew-symmetric bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ on $\Lambda_{\mathcal{D}}$ given by the formula

$$((v_1, \varphi_1), (v_2, \varphi_2))_{\mathcal{D}} = (v_1, v_2) + \varphi_2(v_1) - \varphi_1(v_2).$$

We can apply the construction of Definition 2.5 to these data to get a quantum torus algebra which we denote $\mathcal{D}_{\mathbf{i}}^q$. If we let X_i and B_i denote the generators associated to the basis elements e_i and f_i , respectively, then we have the commutation relations

$$X_i X_j = q^{2\varepsilon_{ij}} X_j X_i, \quad B_i B_j = B_j B_i, \quad X_i B_j = q^{2\delta_{ji}} B_j X_i.$$

We will write $\widehat{\mathcal{D}}_{\mathbf{i}}^q$ for the (noncommutative) fraction field of $\mathcal{D}_{\mathbf{i}}^q$. The following notations will be important in the sequel:

$$\mathbb{B}_k^+ = \prod_{i|(e_k, e_i) > 0} B_i^{(e_k, e_i)}, \quad \mathbb{B}_k^- = \prod_{i|(e_k, e_i) < 0} B_i^{-(e_k, e_i)}, \quad \widehat{X}_i = X_i \prod_j B_j^{(e_i, e_j)}.$$

One can check using the above relations that the elements X_k and \widehat{X}_k commute. Just as before, we have a natural map $\widehat{\mathcal{D}}_{\mathbf{i}'}^q \rightarrow \widehat{\mathcal{D}}_{\mathbf{i}}^q$ whenever \mathbf{i} and \mathbf{i}' are two seeds related by a mutation.

Definition 2.7.

1. The automorphism $\mu_k^{\sharp} : \widehat{\mathcal{D}}_{\mathbf{i}}^q \rightarrow \widehat{\mathcal{D}}_{\mathbf{i}}^q$ is given by

$$\mu_k^{\sharp} = \text{Ad}_{\Psi^q(X_k)/\Psi^q(\widehat{X}_k)}.$$

2. The isomorphism $\mu'_k : \widehat{\mathcal{D}}_{\mathbf{i}'}^q \rightarrow \widehat{\mathcal{D}}_{\mathbf{i}}^q$ is induced by the natural lattice map $\Lambda_{\mathcal{D}, \mathbf{i}'} \rightarrow \Lambda_{\mathcal{D}, \mathbf{i}}$.
3. The mutation map $\mu_k^q : \widehat{\mathcal{D}}_{\mathbf{i}'}^q \rightarrow \widehat{\mathcal{D}}_{\mathbf{i}}^q$ is the composition $\mu_k^q = \mu_k^{\sharp} \circ \mu'_k$.

Notice that the generators X_i span a subalgebra of $\mathcal{D}_{\mathbf{i}}^q$ which is isomorphic to the quantum torus algebra $\mathcal{X}_{\mathbf{i}}^q$ defined previously. Moreover, since X_k and \widehat{X}_k commute, the restriction of μ_k^q to this subalgebra coincides with the mutation map from Definition 2.6.

Although conjugation by $\Psi^q(X_k)/\Psi^q(\widehat{X}_k)$ produces a priori a formal power series, this construction in fact provides a map $\widehat{\mathcal{D}}_{\mathbf{i}'}^q \rightarrow \widehat{\mathcal{D}}_{\mathbf{i}}^q$ of skew fields. Indeed, one has the following formulas for the values of the map μ_k^q on generators.

Theorem 2.8. The map μ_k^q is given on generators by the formulas

$$\mu_k^q(B'_i) = \begin{cases} (qX_k\mathbb{B}_k^+ + \mathbb{B}_k^-)B_k^{-1}(1 + q^{-1}X_k)^{-1} & \text{if } i = k \\ B_i & \text{if } i \neq k \end{cases}$$

and

$$\mu_k^q(X'_i) = \begin{cases} X_i \prod_{p=0}^{|\varepsilon_{ik}|-1} (1 + q^{2p+1}X_k) & \text{if } \varepsilon_{ik} \leq 0 \text{ and } i \neq k \\ X_i X_k^{\varepsilon_{ik}} \prod_{p=0}^{\varepsilon_{ik}-1} (X_k + q^{2p+1})^{-1} & \text{if } \varepsilon_{ik} \geq 0 \text{ and } i \neq k \\ X_k^{-1} & \text{if } i = k. \end{cases}$$

The proof of this theorem can be found in Appendix A.

2.4 The classical limit

We have now seen how to associate, to any seed \mathbf{i} , a noncommutative algebra $\mathcal{D}_{\mathbf{i}}^q$. In addition, we have seen how to associate, to any pair of seeds \mathbf{i} and \mathbf{i}' related by a mutation, an isomorphism $\widehat{\mathcal{D}}_{\mathbf{i}'}^q \rightarrow \widehat{\mathcal{D}}_{\mathbf{i}}^q$ of the corresponding fraction fields.

Note that if we set $q = 1$, then $\mathcal{D}_{\mathbf{i}}^q$ becomes the Laurent polynomial ring $\mathbb{Z}[B_i^{\pm 1}, X_i^{\pm 1}]_{i \in J}$. Its spectrum is a split algebraic torus, which we denote by $\mathcal{D}_{\mathbf{i}}$. The formulas of Theorem 2.8 specialize to

$$\mu_k^* B'_i = \begin{cases} \frac{X_k \prod_{j|\varepsilon_{kj}>0} B_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} B_j^{-\varepsilon_{kj}}}{(1+X_k)B_k} & \text{if } i = k \\ B_i & \text{if } i \neq k \end{cases}$$

and

$$\mu_k^* X'_i = \begin{cases} X_k^{-1} & \text{if } i = k \\ X_i (1 + X_k^{-\text{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & \text{if } i \neq k \end{cases}$$

where $\{B'_j, X'_j\}_{j \in J}$ are the coordinates on $\mathcal{D}_{\mathbf{i}'}$. These formulas define birational maps $\mathcal{D}_{\mathbf{i}} \dashrightarrow \mathcal{D}_{\mathbf{i}'}$ of tori.

Definition 2.9. The *symplectic double* $\mathcal{D} = \mathcal{D}_{|\mathbf{i}|}$ is the scheme over \mathbb{Z} obtained by gluing the split algebraic tori $\mathcal{D}_{\mathbf{i}'}$ for all seeds \mathbf{i}' mutation equivalent to \mathbf{i} using the birational maps above.

In exactly the same way, the algebra $\mathcal{X}_{\mathbf{i}}^q$ becomes the Laurent polynomial ring $\mathbb{Z}[X_i^{\pm 1}]_{i \in J}$ when $q = 1$. Its spectrum is a split algebraic torus denoted $\mathcal{X}_{\mathbf{i}}$. As before, we get a birational map $\mathcal{X}_{\mathbf{i}} \dashrightarrow \mathcal{X}_{\mathbf{i}'}$ whenever two seeds \mathbf{i} and \mathbf{i}' are related by a mutation.

Definition 2.10. The *cluster Poisson variety* $\mathcal{X} = \mathcal{X}_{|\mathbf{i}|}$ is the scheme over \mathbb{Z} obtained by gluing the split algebraic tori $\mathcal{X}_{\mathbf{i}'}$ for all seeds \mathbf{i}' mutation equivalent to \mathbf{i} using the birational maps above.

As the names suggest, the cluster Poisson variety has a natural Poisson structure, and the symplectic double carries a natural symplectic form with a compatible Poisson structure. The cluster Poisson variety embeds into the symplectic double as a Lagrangian subspace [9].

For generic q , we can use the formulas of Theorem 2.8 to glue the algebras $\widehat{\mathcal{X}}_1^q$ and $\widehat{\mathcal{D}}_1^q$:

$$\begin{aligned}\mathcal{X}^q &= \coprod_{i' \in [i]} \widehat{\mathcal{X}}_{i'}^q / \text{identifications}, \\ \mathcal{D}^q &= \coprod_{i' \in [i]} \widehat{\mathcal{D}}_{i'}^q / \text{identifications}.\end{aligned}$$

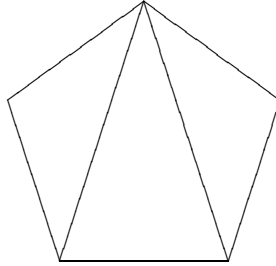
The sets \mathcal{X}^q and \mathcal{D}^q inherit natural algebra structures and in the classical limit $q = 1$ are identified with the function fields of the cluster Poisson variety and symplectic double, respectively. For $q \neq 1$, we think of \mathcal{X}^q and \mathcal{D}^q as the function fields of corresponding “quantum cluster varieties”.

2.5 The disk case

From now on, we will write S for a compact oriented disk with finitely many marked points on its boundary. As we will see in this section, there are quantum cluster varieties naturally associated to such a disk.

Definition 2.11. An *ideal triangulation* of S is a triangulation whose vertices are the marked points on the boundary.

The disk S admits an ideal triangulation if and only if it has at least three marked points on its boundary. From now on, we will always assume this is the case. The illustration below shows an example of an ideal triangulation of a disk with five marked points.



An edge of an ideal triangulation is said to be *external* if it lies along the boundary of S , connecting adjacent marked points, and is said to be *internal* otherwise. For a given ideal triangulation T , we will write $I = I_T$ for the set of edges of T and $J = J_T$ for the set of internal edges.

Definition 2.12. Let T be an ideal triangulation of S . Then we define a skew-symmetric matrix ε_{ij} ($i, j \in I$) by

$$\varepsilon_{ij} = \begin{cases} -1 & \text{if } i, j \text{ share a vertex and } i \text{ is immediately clockwise to } j \\ 1 & \text{if } i, j \text{ share a vertex and } j \text{ is immediately clockwise to } i \\ 0 & \text{otherwise.} \end{cases}$$

Given an ideal triangulation T , we can consider the associated lattice

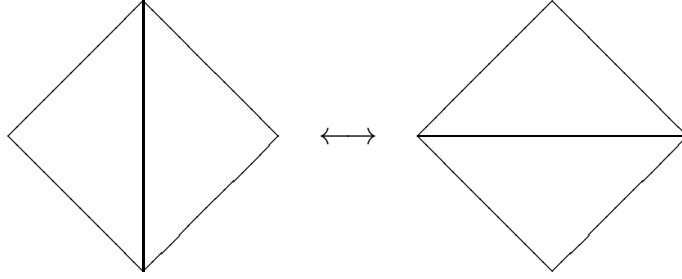
$$\Lambda = \mathbb{Z}[I].$$

This lattice has a basis $\{e_i\}$ given by $e_i = \{i\}$ for $i \in I$ and a \mathbb{Z} -valued skew-symmetric bilinear form (\cdot, \cdot) given on basis elements by

$$(e_i, e_j) = \varepsilon_{ij}.$$

Thus we associate a seed to any ideal triangulation.

Definition 2.13. If k is an internal edge of an ideal triangulation T , then a *flip* at k is the transformation of T that removes k and replaces it by the unique different edge that, together with the remaining edges, forms an ideal triangulation:



It is a fact that any two ideal triangulations are related by a sequence of flips. It is therefore natural to ask how the matrix ε_{ij} changes when we perform a flip at some edge k . To answer this question, first note that there is a natural bijection between the edges of an ideal triangulation and the edges of the triangulation obtained by a flip at some edge. If we use this bijection to identify edges of the flipped triangulation with the set I , then we have the following result.

Proposition 2.14. A flip at an edge k of an ideal triangulation changes the matrix ε_{ij} to the matrix

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } k \in \{i, j\} \\ \varepsilon_{ij} + \frac{|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|}{2} & \text{if } k \notin \{i, j\}. \end{cases}$$

Thus we see that the new matrix ε'_{ij} that we get after performing a flip at the edge k is the same as the matrix that we get by mutating the seed associated to T . It follows that the quantum cluster varieties defined in Section 2 can be *canonically* associated to a disk with marked points so that each seed corresponds to an ideal triangulation of the disk.

Definition 2.15. We will write $\mathcal{D}_{PGL_2, S}^q$ for the algebra \mathcal{D}^q associated to a disk S in this way and $\mathcal{X}_{PGL_2, S}^q$ for the algebra \mathcal{X}^q associated to the disk.

The cluster varieties considered here are related to configurations of flags for the group PGL_2 [6, 10, 2]. This accounts for the notation used in the above definition.

3 The skein algebra and laminations

3.1 Definition of the skein algebra

The quantum cluster algebras that we consider in this paper all arise from the skein algebra of the disk S . Here we will review the the general theory of skein algebras, following [17].

Definition 3.1. By a *curve* in S , we mean an immersion $C \rightarrow S$ of a compact, connected, one-dimensional manifold C with (possibly empty) boundary into S . We require that any boundary points of C map to the marked points on ∂S and no point in the interior of C maps to a marked point. By a *homotopy* of two curves α and β , we mean a homotopy of α and β within the class of such curves. Two curves are said to be *homotopic* if they can be related by homotopy and orientation-reversal.

Definition 3.2. A *multicurve* is an unordered finite set of curves which may contain duplicates. Two multicurves are *homotopic* if there is a bijection between their constituent curves which takes each curve to a homotopic one.

Definition 3.3. A *framed link* in S is a multicurve such that each intersection of strands is transverse and

1. At each crossing, there is an ordering of the strands.
2. At each marked point, there is an equivalence relation on the strands and an ordering on equivalence classes of strands.

By a *homotopy* of framed links, we mean a homotopy through the class of multicurves with transverse intersections where the crossing data are not changed.

When drawing pictures of framed links, we indicate the ordering of strands at a transverse intersection or marked point by making one strand pass “over” the other:



(In the second of these pictures, the dotted line indicates a portion of ∂S containing a marked point.) If two strands are identified by the equivalence relation at a marked point, we indicate this as in the following picture:



Definition 3.4. A multicurve with transverse intersections is said to be *simple* if it has no interior intersections and no contractible curves. Note that any simple multicurve can be regarded as a framed link with the simultaneous ordering at each endpoint.

Definition 3.5. Let us write $\mathcal{K}(S)$ for the module over $\mathbb{Z}[\omega, \omega^{-1}]$ generated by equivalence classes of framed links in S . The *skein module* $\text{Sk}_\omega(S)$ is defined as the quotient of $\mathcal{K}(S)$ by the following local relations. In each of these expressions, we depict the portion of a framed link over a small disk in S . The framed links appearing in a given relation are assumed to be identical to each other outside of the small disk. In the last two relations, the dotted line segment represents a portion of ∂S . In these pictures, there may be additional undrawn curves ending at marked points, provided their order with respect to the drawn curves and each other does not change.

$$\begin{array}{c}
\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \omega^{-2} \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} + \omega^2 \begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array} \\
\\
\bigcirc = -(\omega^4 + \omega^{-4}) \\
\\
\omega \begin{array}{c} \diagup \diagdown \\ \text{---} \end{array} = \begin{array}{c} \diagup \diagdown \\ \text{---} \end{array} = \omega^{-1} \begin{array}{c} \diagdown \diagup \\ \text{---} \end{array} \\
\\
\begin{array}{c} \diagup \diagdown \\ \text{---} \end{array} = \begin{array}{c} \diagup \diagdown \\ \text{---} \end{array} = \begin{array}{c} \diagup \diagdown \\ \text{---} \end{array} = 0
\end{array}$$

If K is a framed link in S , then the class of K in $\text{Sk}_\omega(S)$ will be denoted $[K]$.

Suppose K and L are two links such that the union of the underlying multicurves has transverse intersections. Then the *superposition* $K \cdot L$ is the framed link whose underlying multicurve is the union of the underlying multicurves of K and L , with each strand of K crossing over each strand of L and all other crossings ordered as in K and L .

Proposition 3.6 ([17], Proposition 3.5). $[K \cdot L]$ depends only on the homotopy classes of K and L .

Definition 3.7. For any framed links K and L , choose homotopic links K' and L' such that the union of the multicurves underlying K' and L' is transverse. Then the *superposition product* is defined by

$$[K][L] := [K' \cdot L'].$$

This extends to a product on $\text{Sk}_\omega(S)$ by bilinearity.

Note that the superposition product is well defined and independent of the choice of K' and L' by Proposition 3.6.

3.2 Relation to quantum cluster algebras

We conclude this section by describing the relation, first discovered by Muller in [17], between skein algebras and quantum cluster algebras. A review of the relevant notions from the theory of quantum cluster algebras can be found in Appendix B. As usual, we take S to be a disk with finitely many marked points on its boundary.

Definition 3.8. Let T be an ideal triangulation of S . For any $i \in I$ and $j \in J$, we define

$$b_{ij} = \begin{cases} 1 & \text{if } i, j \text{ share a vertex and } i \text{ is immediately clockwise to } j \\ -1 & \text{if } i, j \text{ share a vertex and } j \text{ is immediately clockwise to } i \\ 0 & \text{otherwise.} \end{cases}$$

These are entries of a skew-symmetric $|I| \times |J|$ matrix which we denote \mathbf{B}_T .

Definition 3.9. Let T be an ideal triangulation of S . For $i, j \in I$, we define

$$\lambda_{ij} = \begin{cases} 1 & \text{if } i, j \text{ share a vertex and } i \text{ is clockwise to } j \\ -1 & \text{if } i, j \text{ share a vertex and } j \text{ is clockwise to } i \\ 0 & \text{otherwise.} \end{cases}$$

These are entries of a skew-symmetric $|I| \times |I|$ matrix which we denote Λ_T . We will use the same notation Λ_T for the associated skew-symmetric bilinear form $\mathbb{Z}^I \times \mathbb{Z}^I \rightarrow \mathbb{Z}$.

Proposition 3.10 ([17], Proposition 7.8). The matrices Λ_T and \mathbf{B}_T satisfy the compatibility condition

$$\sum_k b_{kj} \lambda_{ki} = 4\delta_{ij}.$$

Definition 3.11. Let \mathcal{F} denote the skew-field of fractions of the skein algebra $\text{Sk}_\omega(S)$. For any ideal triangulation T and any vector $v = (v_1, \dots, v_m) \in \mathbb{Z}_{\geq 0}^I$, we will write T^v for the simple multicurve having v_i -many curves homotopic to $i \in I$ and no other components. The corresponding class $[T^v]$ is called a *monomial* in the triangulation T . More generally, we write

$$[T^{u'-u}] = \omega^{-\Lambda_T(u, u')} [T^u]^{-1} [T^{u'}].$$

This is well defined and provides a map $M_T : \mathbb{Z}^I \rightarrow \mathcal{F} - \{0\}$ given by $M_T(v) = [T^v]$.

It is easy to see that the pair (\mathbf{B}_T, M_T) is a quantum seed and Λ_T is the compatibility matrix associated to the toric frame M_T .

Proposition 3.12 ([17], Theorem 7.9). Let T be an ideal triangulation of S , and let T' be the ideal triangulation obtained from T by performing a flip of the edge k . Then the quantum seed $(\mathbf{B}_{T'}, M_{T'})$ is obtained from (\mathbf{B}_T, M_T) by a mutation in the direction k .

It follows from Proposition 3.12 that there is a quantum cluster algebra \mathcal{A} canonically associated to the disk S . This algebra is generated by the cluster variables inside of the skew-field of fractions of the skein algebra.

3.3 Laminations

In order to construct the maps $\mathbb{I}_{\mathcal{A}}^q$ and $\mathbb{I}_{\mathcal{D}}^q$, we need to define the spaces $\mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t)$ and $\mathcal{D}_{PGL_2, S}(\mathbb{Z}^t)$. As explained in the introduction, these can be understood as tropicalizations of a certain moduli spaces associated to the groups SL_2 and PGL_2 . Further explanation can be found in [7, 1, 2, 10].

Definition 3.13. Let S be a disk with finitely many marked points on its boundary. A point of $\mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t)$ is defined as a collection of edges and mutually nonintersecting diagonals of the polygon S with integral weights, subject to the following conditions:

1. The weight of a diagonal is positive.
2. The total weight of the diagonals and edges incident to a given vertex is zero.

Definition 3.14 ([1], Definition 4.1). If S is an oriented disk with finitely many marked points on its boundary, we will write S° for the same disk equipped with the opposite orientation. A point of $\mathcal{D}_{PGL_2, S}(\mathbb{Z}^t)$ is defined as a collection of edges and nonintersecting diagonals of the polygons S and S° with positive integral weights, subject to the following conditions:

1. The total weight of the diagonals and edges incident to a given vertex of S is the same as the total weight of the diagonals and edges incident to the corresponding vertex of S° .
2. The collection does not include an edge of S together with the corresponding edge of S° .

The above definitions are equivalent to the definitions in [7, 1]. In [7], the space $\mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t)$ was understood as a space of measured laminations on S . In [1], the space $\mathcal{D}_{PGL_2, S}(\mathbb{Z}^t)$ was understood as a space of measured laminations on the surface $S_{\mathcal{D}}$ obtained by gluing S and S° along corresponding boundary edges.

4 Duality map for the quantum Poisson variety

4.1 Realization of the cluster Poisson variety

We have associated a quantum cluster algebra \mathcal{A} to the disk S . Let \mathcal{F} be the ambient skew-field of this algebra \mathcal{A} .

Definition 4.1. Let T be an ideal triangulation of S . For any $j \in J$, we define an element $X_j = X_{j;T}$ of \mathcal{F} by the formula

$$X_j = M_T \left(\sum_s \varepsilon_{js} \mathbf{e}_s \right)$$

where the \mathbf{e}_s are standard basis vectors. In addition, we will use the notation $q = \omega^4$.

As the notation suggests, these elements are related to the generators of the quantum Poisson variety.

Proposition 4.2. The elements X_j ($j \in J$) satisfy the commutation relations

$$X_i X_j = q^{2\varepsilon_{ij}} X_j X_i.$$

Proof. By the properties of toric frames, we have

$$X_i X_j = \omega^{-2\Lambda_T(\sum_s \varepsilon_{is} \mathbf{e}_s, \sum_t \varepsilon_{jt} \mathbf{e}_t)} X_j X_i.$$

Now the compatibility condition in the definition of a quantum seed implies

$$\begin{aligned} \Lambda_T\left(\sum_s \varepsilon_{is} \mathbf{e}_s, \sum_t \varepsilon_{jt} \mathbf{e}_t\right) &= \sum_{s,t} \varepsilon_{is} \varepsilon_{jt} \Lambda_T(\mathbf{e}_s, \mathbf{e}_t) \\ &= \sum_{s,t} \varepsilon_{is} \varepsilon_{jt} \lambda_{st} \\ &= - \sum_s \varepsilon_{is} \sum_t b_{tj} \lambda_{ts} \\ &= -4\varepsilon_{ij}. \end{aligned}$$

Therefore $X_i X_j = \omega^{8\varepsilon_{ij}} X_j X_i = q^{2\varepsilon_{ij}} X_j X_i$. □

Proposition 4.3. For each i , let $X'_i = X_{i,T'}$. Then

$$X'_i = \begin{cases} X_i \prod_{p=0}^{|\varepsilon_{ik}|-1} (1 + q^{2p+1} X_k) & \text{if } \varepsilon_{ik} \leq 0 \text{ and } i \neq k \\ X_i X_k^{\varepsilon_{ik}} \prod_{p=0}^{\varepsilon_{ik}-1} (X_k + q^{2p+1})^{-1} & \text{if } \varepsilon_{ik} \geq 0 \text{ and } i \neq k \\ X_k^{-1} & \text{if } i = k. \end{cases}$$

Proof. According to Lemma 5.4 of [20], a mutation in the direction k transforms X_i to the new element

$$X'_i = \begin{cases} X_i \prod_{p=0}^{|b_{ki}|-1} (1 + \omega^{-2(-d_k p - \frac{d_k}{2})} X_k) & \text{if } b_{ki} \leq 0 \text{ and } i \neq k \\ X_i X_k^{b_{ki}} \prod_{p=0}^{b_{ki}-1} (X_k + \omega^{-2(-d_k p - \frac{d_k}{2})})^{-1} & \text{if } b_{ki} \geq 0 \text{ and } i \neq k \\ X_k^{-1} & \text{if } i = k \end{cases}$$

where d_k is the number appearing in Definition B.2. The statement follows if we substitute $d_k = 4$ into this formula. □

4.2 Construction of the map

We will now define the map $\mathbb{I}_{\mathcal{A}}^q$. To do this, let l be a point of $\mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t)$, represented by a collection of curves on S . By the definition of the space $\mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t)$, these curves are

nonintersecting, so there exists an ideal triangulation T_l of S such that each of these curves coincides with an edge of T_l . Let

$$\mathbf{w} = (w_1, \dots, w_m)$$

be the integral vector whose i th component w_i is the weight of the curve corresponding to the edge i of T_l .

Definition 4.4. We will write $\mathbb{I}_{\mathcal{A}}^q(l)$ for the element of \mathcal{F} given by

$$\mathbb{I}_{\mathcal{A}}^q(l) = M_{T_l}(\mathbf{w}).$$

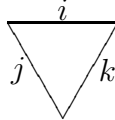
It is clear from the definition of the toric frames that this is independent of the choice of T_l . Our goal is to show that this definition provides a map $\mathbb{I}_{\mathcal{A}}^q : \mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t) \rightarrow \mathcal{X}_{PGL_2, S}^q$. To prove this, we will need a technical lemma on the \mathbf{g} -vectors from Section B.2.

Lemma 4.5. Fix an ideal triangulation T of S , and let c_1, \dots, c_{2k} be noncontractible curves connecting marked points. Assume these curves form a closed path on S in which c_i and c_{i+1} share a common endpoint for $i = 1, \dots, 2k$ where we number the indices modulo $2k$. Let \mathbf{g}_{c_i} be the \mathbf{g} -vector associated to c_i . Then the alternating sum

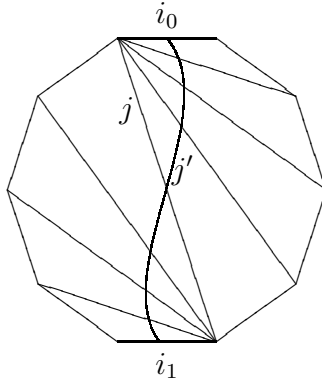
$$\mathbf{s} = \sum_{i=1}^{2k} (-1)^i \mathbf{g}_{c_i}$$

is an integral linear combination of expressions of the form $\sum_s \varepsilon_{is} \mathbf{e}_s$.

Proof. Consider the triangle in T that includes $i \in I - J$ as one of its edges and label the other edges as follows:



We can define a vector associated to this edge i as $\mathbf{y}_i = \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_k$. Fix an edge $j \in J$. For any endpoint v of j , there is a collection of edges of T that start at v and lie in the counterclockwise direction from j . Consider an arc j' that goes diagonally across j , intersecting each of these edges transversally and terminating on the boundary. An example is illustrated below.



Given such an arc j' , let E_j be the set of all edges in T that j' crosses. Then we can form the sum

$$\mathbf{p}_j = \mathbf{y}_{i_0} + \sum_{i \in E_j} \mathbf{x}_i + \mathbf{y}_{i_1}$$

where i_0 and i_1 are the edges on which j' terminates and we have written $\mathbf{x}_i = \sum_s \varepsilon_{is} \mathbf{e}_s$. One can check that this expression \mathbf{p}_j equals $2\mathbf{e}_j$. Now let c be a curve connecting marked points. It follows from the results of [18] that the \mathbf{g} -vector \mathbf{g}_c is an alternating sum of standard basis vectors corresponding to a path in T . In fact, the entire sum appearing in the statement of the lemma can be written as an alternating sum $\mathbf{s} = \sum_{l=1}^{2N} (-1)^l \mathbf{e}_{j_l}$ of standard basis vectors associated to the edges j_1, \dots, j_{2N} of a closed path in T . We assume that these edges are ordered consecutively. To each edge j of this path, we associate the arc j' described above. We have

$$2\mathbf{s} = 2 \sum_{l=1}^{2N} (-1)^l \mathbf{e}_{j_l} = \sum_{l=1}^{2N} (-1)^l \mathbf{p}_{j_l}$$

and all of the \mathbf{y} -terms cancel on the right hand side. Thus this expression is linear combination of the \mathbf{x}_k . By construction, these \mathbf{x}_k are associated to the points of intersection between edges of T and a certain closed path on S . Each edge intersects this path in an even number of points, so the vector $2\mathbf{s}$ is in fact a linear combination of the \mathbf{x}_k with *even* coefficients. Dividing by 2 yields the desired result. \square

Theorem 4.6. Let T be an ideal triangulation of S , and let $X_j = X_{j;T}$ be defined as above. Then for any $l \in \mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t)$, the element $\mathbb{I}_{\mathcal{A}}^q(l)$ is a Laurent polynomial in the X_j with coefficients in $\mathbb{Z}_{\geq 0}[q, q^{-1}]$.

Proof. By applying the relations from Definition 3.5, we can write

$$\mathbb{I}_{\mathcal{A}}^q(l) = \omega^\alpha \prod_{i=1}^m M_{T_i}(\mathbf{e}_i)^{w_i}$$

where $\alpha = \sum_{i < j} \Lambda_{T_i}(\mathbf{e}_i, \mathbf{e}_j) w_i w_j$. Then Corollary B.14 implies

$$\mathbb{I}_{\mathcal{A}}^q(l) = \omega^\alpha \prod_{i=1}^m (F_i \cdot M_T(\mathbf{g}_i))^{w_i}.$$

Each F_i in this last line denotes a polynomial in the expressions $M_T(\sum_j \varepsilon_{kj} \mathbf{e}_j)$ with coefficients in $\mathbb{Z}_{\geq 0}[\omega^4, \omega^{-4}]$. The \mathbf{g}_i are integral vectors. We have

$$\Lambda_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i, \mathbf{e}_j\right) = \sum_i \varepsilon_{ki} \lambda_{ij} = \sum_i b_{ik} \lambda_{ij} = 4\delta_{kj}$$

so that

$$M_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right) M_T(\mathbf{e}_j) = \omega^{-8\delta_{kj}} M_T(\mathbf{e}_j) M_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right).$$

It follows that we can rearrange the factors in the last expression for $\mathbb{I}_{\mathcal{A}}^q(l)$ to get

$$\mathbb{I}_{\mathcal{A}}^q(l) = \omega^\alpha Q \cdot \prod_{i=1}^m M_T(\mathbf{g}_i)^{w_i}$$

where Q is a polynomial in the expressions $M_T(\sum_j \varepsilon_{kj} \mathbf{e}_j)$ with coefficients in the semiring $\mathbb{Z}_{\geq 0}[\omega^4, \omega^{-4}]$. By Lemma 6.2 in [20], we know that $\Lambda_{T_l}(\mathbf{e}_i, \mathbf{e}_j) = \Lambda_T(\mathbf{g}_i, \mathbf{g}_j)$, and so

$$\begin{aligned} M_T\left(\sum_i w_i \mathbf{g}_i\right) &= \omega^{\sum_{i < j} \Lambda_T(\mathbf{g}_i, \mathbf{g}_j) w_i w_j} \prod_{i=1}^m M_T(\mathbf{g}_i)^{w_i} \\ &= \omega^\alpha \prod_{i=1}^m M_T(\mathbf{g}_i)^{w_i}. \end{aligned}$$

Hence

$$\mathbb{I}_{\mathcal{A}}^q(l) = Q \cdot M_T\left(\sum_i w_i \mathbf{g}_i\right).$$

The vector $\sum_i w_i \mathbf{g}_i$ is a sum of vectors of the type appearing in the statement of Lemma 4.5, so this lemma implies that the second factor is a monomial in the X_j with coefficient in $\mathbb{Z}_{\geq 0}[\omega^4, \omega^{-4}]$. This completes the proof. \square

By the properties established in Propositions 4.2, and 4.3, we can regard the Laurent polynomial of Theorem 4.6 as an element of the algebra $\mathcal{X}_{PGL_2, S}^q$. Thus we have constructed a canonical map $\mathbb{I}_{\mathcal{A}}^q : \mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t) \rightarrow \mathcal{X}_{PGL_2, S}^q$ as desired.

Interestingly, the proof of this theorem uses several tools that were not present in [3], including the theory of quantum cluster algebras [4, 20] and the work of Muller [17]. In [3], the construction was instead based on the “quantum trace map” of Bonahon and Wong [5]. Recent work of Lê [15] suggests that these two approaches are in fact equivalent.

4.3 Properties

We will now prove several properties of the map $\mathbb{I}_{\mathcal{A}}^q$ conjectured in [6, 8].

Theorem 4.7. The map $\mathbb{I}_{\mathcal{A}}^q : \mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t) \rightarrow \mathcal{X}_{PGL_2, S}^q$ satisfies the following properties:

1. Each $\mathbb{I}_{\mathcal{A}}^q(l)$ is a Laurent polynomial in the variables X_i with coefficients in $\mathbb{Z}_{\geq 0}[q, q^{-1}]$.
2. The expression $\mathbb{I}_{\mathcal{A}}^q(l)$ agrees with the Laurent polynomial $\mathbb{I}_{\mathcal{A}}(l)$ when $q = 1$.
3. Let $*$ be the canonical involutive antiautomorphism of \mathcal{X}^q that fixes each X_i and sends q to q^{-1} . Then $*\mathbb{I}_{\mathcal{A}}^q(l) = \mathbb{I}_{\mathcal{A}}^q(l)$.
4. The highest term of $\mathbb{I}_{\mathcal{A}}^q(l)$ is

$$q^{-\sum_{i < j} \varepsilon_{ij} a_i a_j} X_1^{a_1} \dots X_n^{a_n}$$

where $X_1^{a_1} \dots X_n^{a_n}$ is the highest term of the classical expression $\mathbb{I}_{\mathcal{A}}(l)$.

5. For any $l, l' \in \mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t)$, we have

$$\mathbb{I}_{\mathcal{A}}^q(l) \mathbb{I}_{\mathcal{A}}^q(l') = \sum_{l'' \in \mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t)} c^q(l, l'; l'') \mathbb{I}_{\mathcal{A}}^q(l'')$$

where $c^q(l, l'; l'') \in \mathbb{Z}[q, q^{-1}]$ and only finitely many terms are nonzero.

Proof. 1. This was proved in Theorem 4.6.

2. This follows immediately from the definition of $\mathbb{I}_{\mathcal{A}}$ given in [7].

3. If K is any framed link in S , let K^\dagger be the framed link with the same underlying multicurve and the order of each crossing reversed. By Proposition 3.11 of [17] the map sending $[K]$ to $[K^\dagger]$ and ω to $\omega^\dagger := \omega^{-1}$ extends to an involutive antiautomorphism of the skein algebra $\text{Sk}_\omega(S)$. It extends further to an antiautomorphism of the fraction field \mathcal{F} by the rule $(xy^{-1})^\dagger = (y^\dagger)^{-1}x^\dagger$. This operation is compatible with $*$ in the sense that

$$X_j^\dagger = X_j = *X_j$$

and

$$q^\dagger = q^{-1} = *q.$$

It is easy to check that \dagger preserves $M_{T_l}(\mathbf{w}) = \omega^{\sum_{i < j} \Lambda_{T_l}(\mathbf{e}_i, \mathbf{e}_j)} M_{T_l}(\mathbf{e}_1)^{w_1} \dots M_{T_l}(\mathbf{e}_n)^{w_m}$. It follows that $\mathbb{I}_{\mathcal{A}}^q(l)$ is $*$ -invariant.

4. By part 1, we can write

$$\mathbb{I}_{\mathcal{A}}^q(l) = \sum_{(i_1, \dots, i_n) \in \text{supp}(l)} c_{i_1, \dots, i_n} X_1^{i_1} \dots X_n^{i_n}$$

for some finite subset $\text{supp}(l) \subseteq \mathbb{Z}^n$ where the coefficient $c_{i_1, \dots, i_n} \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$ is nonzero for all $(i_1, \dots, i_n) \in \text{supp}(l)$. Consider the coefficient $c = c_{a_1, \dots, a_n}$. We can expand this coefficient as $c = \sum_s c_s q^s$. By setting $q = 1$, we see that $\sum_s c_s = 1$. It follows that we must have $c_s = 1$ for some s , and all other terms must vanish. Thus $c = q^s$, and the leading term of $\mathbb{I}_{\mathcal{A}}^q(l)$ has the form

$$q^s X_1^{a_1} \dots X_n^{a_n}$$

for some integer s . By part 3, we know that the expression $\mathbb{I}_{\mathcal{A}}^q(l)$, and hence its leading term, is $*$ -invariant. This means

$$q^{-s} X_n^{a_n} \dots X_1^{a_1} = q^s X_1^{a_1} \dots X_n^{a_n}.$$

We have

$$\begin{aligned} X_n^{a_n} \dots X_1^{a_1} &= q^{-2 \sum_{j=2}^n \varepsilon_{1j} a_1 a_j} X_1^{a_1} (X_n^{a_n} \dots X_2^{a_2}) \\ &= q^{-2 \sum_{j=2}^n \varepsilon_{1j} a_1 a_j} q^{-2 \sum_{j=3}^n \varepsilon_{2j} a_2 a_j} X_1^{a_1} X_2^{a_2} (X_n^{a_n} \dots X_3^{a_3}) \\ &= \dots \\ &= q^{-2 \sum_{i < j} \varepsilon_{ij} a_i a_j} X_1^{a_1} \dots X_n^{a_n}, \end{aligned}$$

so $*$ -invariance of the leading term is equivalent to

$$q^{-s-2\sum_{i<j}\varepsilon_{ij}a_ia_j}X_1^{a_1}\dots X_n^{a_n}=q^sX_1^{a_1}\dots X_n^{a_n}.$$

Equating coefficients, we see that $s = -\sum_{i<j}\varepsilon_{ij}a_ia_j$. Thus we see that the highest term equals $q^{-\sum_{i<j}\varepsilon_{ij}a_ia_j}X_1^{a_1}\dots X_n^{a_n}$.

5. Let $l, l' \in \mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t)$. Then there exist ideal triangulations T_l and $T_{l'}$ of S and integral vectors \mathbf{w} and \mathbf{w}' such that $\mathbb{I}_{\mathcal{A}}^q(l) = M_{T_l}(\mathbf{w})$ and $\mathbb{I}_{\mathcal{A}}^q(l') = M_{T_{l'}}(\mathbf{w}')$. For any integral vector $\mathbf{v} = (v_1, \dots, v_m)$, we can consider the vector \mathbf{v}_+ whose i th component equals v_i if $v_i \geq 0$ and zero otherwise. We can likewise consider the vector \mathbf{v}_- whose i th component equals v_i if $v_i \leq 0$ and zero otherwise. Then there exists an integer N such that

$$\begin{aligned}\mathbb{I}_{\mathcal{A}}^q(l)\mathbb{I}_{\mathcal{A}}^q(l') &= M_{T_l}(\mathbf{w})M_{T_{l'}}(\mathbf{w}') \\ &= \omega^N M_{T_l}(\mathbf{w}_-)M_{T_l}(\mathbf{w}_+)M_{T_{l'}}(\mathbf{w}'_+)M_{T_{l'}}(\mathbf{w}'_-) \\ &= \omega^N M_{T_l}(\mathbf{w}_-)[T_l^{\mathbf{w}+}][T_{l'}^{\mathbf{w}'+}]M_{T_{l'}}(\mathbf{w}'_-).\end{aligned}$$

By applying the skein relations, we can write the product $[T_l^{\mathbf{w}+}][T_{l'}^{\mathbf{w}'+}]$ as

$$[T_l^{\mathbf{w}+}][T_{l'}^{\mathbf{w}'+}] = \sum_{i=1}^k d_i^{\omega} [K_i]$$

where the K_i is are distinct simple multicurves on S and $d_i^{\omega} \in \mathbb{Z}[\omega, \omega^{-1}]$ are nonzero. Therefore

$$\mathbb{I}_{\mathcal{A}}^q(l)\mathbb{I}_{\mathcal{A}}^q(l') = \sum_{i=1}^k \omega^N d_i^{\omega} M_{T_i}(\mathbf{w}_-)M_{T_i}(\mathbf{v}_i)M_{T_i}(\mathbf{w}'_-)$$

where T_i is an ideal triangulation of S such that each each curve of K_i coincides with an edge of T_i . Here we have written \mathbf{v}_i for the unique integral vector such that $[K_i] = M_{T_i}(\mathbf{v}_i)$. Each of the products $\omega^N d_i^{\omega} M_{T_i}(\mathbf{w}_-)M_{T_i}(\mathbf{v}_i)M_{T_i}(\mathbf{w}'_-)$ in this last expression can be written as $c^{\omega}(l, l'; l_i)\mathbb{I}_{\mathcal{A}}^q(l_i)$ for some $l_i \in \mathcal{A}_{SL_2, S}^0(\mathbb{Z}^t)$ and some $c^{\omega}(l, l'; l_i) \in \mathbb{Z}[\omega, \omega^{-1}]$. Hence

$$\mathbb{I}_{\mathcal{A}}^q(l)\mathbb{I}_{\mathcal{A}}^q(l') = \sum_{i=1}^k c^{\omega}(l, l'; l_i)\mathbb{I}_{\mathcal{A}}^q(l_i).$$

The left hand side of this last equation is a Laurent polynomial with coefficients in $\mathbb{Z}[q, q^{-1}]$. Let us write $[\mathbb{I}_{\mathcal{A}}^q(l_i)]^H$ for the highest term of $\mathbb{I}_{\mathcal{A}}^q(l_i)$. We can impose a lexicographic total ordering \geq on the set of all commutative Laurent monomials in X_1, \dots, X_n so that $[\mathbb{I}_{\mathcal{A}}^1(l_i)]^H$ is indeed the highest term of $\mathbb{I}_{\mathcal{A}}^1(l_i)$ with respect to this total ordering. These highest terms are distinct, so we may assume

$$[\mathbb{I}_{\mathcal{A}}^1(l_1)]^H > [\mathbb{I}_{\mathcal{A}}^1(l_2)]^H > \dots > [\mathbb{I}_{\mathcal{A}}^1(l_k)]^H.$$

Consider the expression $c^\omega(l, l'; l_1)[\mathbb{I}_{\mathcal{A}}^q(l_1)]^H$. It cannot cancel with any other term in the sum, so we must have $c^\omega(l, l'; l_1) \in \mathbb{Z}[q, q^{-1}]$. It follows that the sum

$$\sum_{i=2}^k c^\omega(l, l'; l_i) \mathbb{I}_{\mathcal{A}}^q(l_i)$$

is a Laurent polynomial with coefficients in $\mathbb{Z}[q, q^{-1}]$. Arguing as before, we see that $c^\omega(l, l'; l_2) \in \mathbb{Z}[q, q^{-1}]$. Continuing in this way, we see that all $c^\omega(l, l'; l_i)$ lie in $\mathbb{Z}[q, q^{-1}]$. This completes the proof. \square

In [6, 8], Fock and Goncharov conjectured in addition that the Laurent polynomials $c^q(l, l', l'')$ have *positive* coefficients. This property should be closely related to Conjecture 4.20 of [19].

5 Duality map for the quantum symplectic double

5.1 Realization of the symplectic double

We have associated a quantum cluster algebra \mathcal{A} to the disk S . One can likewise associate a quantum cluster algebra \mathcal{A}° to the disk S° obtained from S by reversing its orientation. If T is any ideal triangulation of S , then there is a corresponding triangulation T° of S° . We will write Λ_{T° and M_{T° respectively for the compatibility matrix and toric frame corresponding to the triangulation T° . We will write \mathbf{e}_i for the standard basis vector in \mathbb{Z}^m associated to the edge i of T or the corresponding edge of T° . Note that we have

$$\Lambda_{T^\circ}(\mathbf{e}_i, \mathbf{e}_j) = -\Lambda_T(\mathbf{e}_i, \mathbf{e}_j)$$

for all i and j . Let \mathcal{F} be the ambient skew-field of the quantum cluster algebra \mathcal{A} , and let \mathcal{F}° be the skew-field of the quantum cluster algebra \mathcal{A}° . Each of these skew-fields is a two-sided module over the ring $\mathbb{Z}[\omega, \omega^{-1}]$, so we can form the algebra

$$\mathcal{F}_{\mathcal{D}} = (\mathcal{F} \otimes_{\mathbb{Z}[\omega, \omega^{-1}]} \mathcal{F}^\circ) / \mathcal{I}$$

where \mathcal{I} is the two-sided ideal generated by elements of the form $M_T(-\mathbf{e}_i) \otimes M_{T^\circ}(\mathbf{e}_i) - 1$ for $i \in I - J$.

Definition 5.1. Let T be an ideal triangulation of S . For any $j \in J$, we define elements $B_j = B_{j;T}$ and $X_j = X_{j;T}$ of $\mathcal{F}_{\mathcal{D}}$ by the formulas

$$\begin{aligned} X_j &= M_T\left(\sum_s \varepsilon_{js} \mathbf{e}_s\right) \otimes 1, \\ B_j &= M_T(-\mathbf{e}_j) \otimes M_{T^\circ}(\mathbf{e}_j). \end{aligned}$$

In addition, we will use the notation $q = \omega^4$.

The following result relates these elements to the generators appearing in the definition of the quantum symplectic double.

Proposition 5.2. The elements B_j and X_j ($j \in J$) satisfy the following commutation relations:

$$X_i X_j = q^{2\varepsilon_{ij}} X_j X_i, \quad B_i B_j = B_j B_i, \quad X_i B_j = q^{2\delta_{ij}} B_j X_i.$$

Proof. The proof of the first relation is identical to the proof of Proposition 4.2. To prove the second relation, observe that

$$\begin{aligned} B_i B_j &= M_T(-\mathbf{e}_i) M_T(-\mathbf{e}_j) \otimes M_{T^\circ}(\mathbf{e}_i) M_{T^\circ}(\mathbf{e}_j) \\ &= \omega^{-2(\Lambda_T(\mathbf{e}_i, \mathbf{e}_j) + \Lambda_{T^\circ}(\mathbf{e}_i, \mathbf{e}_j))} M_T(-\mathbf{e}_j) M_T(-\mathbf{e}_i) \otimes M_{T^\circ}(\mathbf{e}_j) M_{T^\circ}(\mathbf{e}_i) \\ &= \omega^{-2(\Lambda_T(\mathbf{e}_i, \mathbf{e}_j) + \Lambda_{T^\circ}(\mathbf{e}_i, \mathbf{e}_j))} B_j B_i \end{aligned}$$

and

$$\Lambda_T(\mathbf{e}_i, \mathbf{e}_j) + \Lambda_{T^\circ}(\mathbf{e}_i, \mathbf{e}_j) = 0.$$

Therefore $B_i B_j = B_j B_i$. Finally, we have

$$\begin{aligned} X_i B_j &= M_T\left(\sum_s \varepsilon_{is} \mathbf{e}_s\right) M_T(-\mathbf{e}_j) \otimes M_{T^\circ}(\mathbf{e}_j) \\ &= \omega^{-2\Lambda_T(\sum_s \varepsilon_{is} \mathbf{e}_s, -\mathbf{e}_j)} M_T(-\mathbf{e}_j) M_T\left(\sum_s \varepsilon_{is} \mathbf{e}_s\right) \otimes M_{T^\circ}(\mathbf{e}_j) \\ &= \omega^{-2\Lambda_T(\sum_s \varepsilon_{is} \mathbf{e}_s, -\mathbf{e}_j)} B_j X_i. \end{aligned}$$

By the compatibility condition, we have

$$\begin{aligned} \Lambda_T\left(\sum_s \varepsilon_{is} \mathbf{e}_s, -\mathbf{e}_j\right) &= -\sum_s \varepsilon_{is} \Lambda_T(\mathbf{e}_s, \mathbf{e}_j) \\ &= -\sum_s b_{si} \lambda_{sj} \\ &= -4\delta_{ij}. \end{aligned}$$

Therefore $X_i B_j = \omega^{8\delta_{ij}} B_j X_i = q^{2\delta_{ij}} B_j X_i$. □

In the following result, T' denotes the triangulation obtained from T by a flip of the edge k .

Proposition 5.3. For each i , let $B'_i = B_{i, T'}$. Then

$$B'_i = \begin{cases} (qX_k \mathbb{B}_k^+ + \mathbb{B}_k^-) B_k^{-1} (1 + q^{-1} X_k)^{-1} & \text{if } i = k \\ B_i & \text{if } i \neq k. \end{cases}$$

Proof. By Proposition B.10, we have

$$\begin{aligned}
M_{T'}(\mathbf{e}_k) &= M_T\left(-\mathbf{e}_k + \sum_i [\varepsilon_{ki}]_+ \mathbf{e}_i\right) + M_T\left(-\mathbf{e}_k + \sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) \\
&= M_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i - \mathbf{e}_k + \sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) + M_T\left(-\mathbf{e}_k + \sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) \\
&= \omega^\alpha M_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right) M_T(-\mathbf{e}_k) M_T\left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) \\
&\quad + \omega^\beta M_T(-\mathbf{e}_k) M_T\left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right)
\end{aligned}$$

where we have written

$$\alpha = \Lambda_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i - \mathbf{e}_k, \sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) + \Lambda_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i, -\mathbf{e}_k\right)$$

and

$$\beta = \Lambda_T\left(-\mathbf{e}_k, \sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right).$$

Factoring, we obtain

$$M_{T'}(\mathbf{e}_k) = \left(1 + \omega^{\alpha-\beta} M_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right)\right) \omega^\beta M_T(-\mathbf{e}_k) M_T\left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right).$$

A straightforward calculation using the compatibility condition shows $\alpha - \beta = -4$. Substituting this into the last expression, we see that

$$M_{T'}(\mathbf{e}_k) \otimes 1 = (1 + q^{-1} X_k) \omega^\beta (M(-\mathbf{e}_k) \otimes 1) \left(M_T\left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) \otimes 1\right).$$

On the other hand, we have

$$\begin{aligned}
1 \otimes M_{(T')^\circ}(\mathbf{e}_k) &= 1 \otimes M_{T^\circ}\left(-\mathbf{e}_k + \sum_i [\varepsilon_{ki}]_+ \mathbf{e}_i\right) + 1 \otimes M_{T^\circ}\left(-\mathbf{e}_k + \sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) \\
&= M_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i - \sum_i [\varepsilon_{ki}]_+ \mathbf{e}_i + \sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) \otimes M_{T^\circ}\left(-\mathbf{e}_k + \sum_i [\varepsilon_{ki}]_+ \mathbf{e}_i\right) \\
&\quad + M_T\left(-\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i + \sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) \otimes M_{T^\circ}\left(-\mathbf{e}_k + \sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right).
\end{aligned}$$

By the properties of toric frames, this equals

$$\begin{aligned}
&\omega^\gamma M_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right) M_T\left(-\sum_i [\varepsilon_{ki}]_+ \mathbf{e}_i\right) M_T\left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) \otimes M_{T^\circ}\left(\sum_i [\varepsilon_{ki}]_+ \mathbf{e}_i\right) M_{T^\circ}(-\mathbf{e}_k) \\
&\quad + \omega^\delta M_T\left(-\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) M_T\left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) \otimes M_{T^\circ}\left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i\right) M_{T^\circ}(-\mathbf{e}_k)
\end{aligned}$$

for certain exponents γ and δ . Factoring, we obtain

$$\begin{aligned}
& \left(\omega^{\gamma-\delta} M_T \left(\sum_i \varepsilon_{ki} \mathbf{e}_i \right) M_T \left(- \sum_i [\varepsilon_{ki}]_+ \mathbf{e}_i \right) \otimes M_{T^\circ} \left(\sum_i [\varepsilon_{ki}]_+ \mathbf{e}_i \right) \right. \\
& \quad \left. + M_T \left(- \sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i \right) \otimes M_{T^\circ} \left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i \right) \right) \omega^\delta M_T \left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i \right) \otimes M_{T^\circ}(-\mathbf{e}_k) \\
& = (\omega^{\gamma-\delta} X_k \mathbb{B}_k^+ + \mathbb{B}_k^-) \omega^\delta (1 \otimes M_{T^\circ}(-\mathbf{e}_k)) \left(M_T \left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i \right) \otimes 1 \right).
\end{aligned}$$

A straightforward calculation shows that $\delta = \beta$ and $\gamma - \delta = 4$. Substituting these into the last expression, we see that

$$1 \otimes M_{(T')^\circ}(\mathbf{e}_k) = (qX_k \mathbb{B}_k^+ + \mathbb{B}_k^-) \omega^\beta (1 \otimes M_{T^\circ}(-\mathbf{e}_k)) \left(M_T \left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i \right) \otimes 1 \right).$$

Finally, by the definition of B'_k , we have

$$\begin{aligned}
B'_k &= M_{T'}(-\mathbf{e}_k) \otimes M_{(T')^\circ}(\mathbf{e}_k) = (1 \otimes M_{(T')^\circ}(\mathbf{e}_k)) (M_{T'}(\mathbf{e}_k) \otimes 1)^{-1} \\
&= (qX_k \mathbb{B}_k^+ + \mathbb{B}_k^-) \omega^\beta (1 \otimes M_{T^\circ}(-\mathbf{e}_k)) \left(M_T \left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i \right) \otimes 1 \right) \\
&\quad \cdot \left(M_T \left(\sum_i [-\varepsilon_{ki}]_+ \mathbf{e}_i \right) \otimes 1 \right)^{-1} (M_T(-\mathbf{e}_k) \otimes 1)^{-1} \omega^{-\beta} (1 + q^{-1} X_k)^{-1} \\
&= (qX_k \mathbb{B}_k^+ + \mathbb{B}_k^-) B_k^{-1} (1 + q^{-1} X_k)^{-1}.
\end{aligned}$$

For $i \neq k$, we clearly have $B'_i = B_i$. This completes the proof. \square

By Proposition 4.3, there is a similar statement describing the transformation of the X_j under a flip of the triangulation.

5.2 Construction of the map

We are now ready to define the map $\mathbb{I}_{\mathcal{D}}^q$. To do this, let l be a point of $\mathcal{D}_{PGL_2, S}(\mathbb{Z}^t)$, represented by a collection of curves on S and S° . Write \mathcal{C} for the collection of curves on S and \mathcal{C}° for the collection of curves on S° . There is an ideal triangulation $T_{\mathcal{C}}$ of S such that each curve in \mathcal{C} coincides with an edge of $T_{\mathcal{C}}$, and there is an ideal triangulation $T_{\mathcal{C}^\circ}$ of S° such that each curve in \mathcal{C}° coincides with an edge of $T_{\mathcal{C}^\circ}$. Fix an ideal triangulation T of S . Then each $c \in \mathcal{C}$ determines an integral \mathbf{g} -vector $\mathbf{g}_c = \mathbf{g}_{c, T_{\mathcal{C}}}^T$. The triangulation T determines a corresponding ideal triangulation of S° , and so we can likewise associate to each $c^\circ \in \mathcal{C}^\circ$ an integral vector $\mathbf{g}_{c^\circ} = \mathbf{g}_{c^\circ, T_{\mathcal{C}^\circ}}^T$.

Lemma 5.4. The number

$$N_l := \Lambda_T(\mathbf{g}_{\mathcal{C}}, \mathbf{g}_{\mathcal{C}^\circ}),$$

where

$$\mathbf{g}_{\mathcal{C}} = \sum_{c \in \mathcal{C}} \mathbf{g}_c, \quad \mathbf{g}_{\mathcal{C}^\circ} = \sum_{c^\circ \in \mathcal{C}^\circ} \mathbf{g}_{c^\circ},$$

is independent of the triangulation T .

Proof. Let T and T' be two different triangulations. Let $\mathbf{g}_{c,T}^T$ and $\mathbf{g}_{c^\circ,T}^T$ be the \mathbf{g} -vectors associated to the arcs $c \in \mathcal{C}$ and $c^\circ \in \mathcal{C}^\circ$, respectively, defined using the triangulation T . Similarly, let $\mathbf{g}_{c,T'}^{T'}$ and $\mathbf{g}_{c^\circ,T'}^{T'}$ be the vectors defined using the triangulation T' . Number the edges of T so that each edge is identified with some number $1, \dots, m$. Using the recurrence relations discussed in Section 1 of [20], we can write

$$\mathbf{g}_{c,T}^T = P_{c,T}^T(\mathbf{e}_1, \dots, \mathbf{e}_m), \quad \mathbf{g}_{c^\circ,T}^T = P_{c^\circ,T}^T(\mathbf{e}_1, \dots, \mathbf{e}_m)$$

for linear polynomials $P_{c,T}^T$ and $P_{c^\circ,T}^T$, and we also have

$$\mathbf{g}_{c,T'}^{T'} = P_{c,T'}^{T'}(\mathbf{g}_{1,T}^{T'}, \dots, \mathbf{g}_{m,T}^{T'}), \quad \mathbf{g}_{c^\circ,T'}^{T'} = P_{c^\circ,T'}^{T'}(\mathbf{g}_{1,T}^{T'}, \dots, \mathbf{g}_{m,T}^{T'})$$

where $\mathbf{g}_{i,T}^{T'}$ is the \mathbf{g} -vector associated to the i th edge of T and the triangulation T' . There exist coefficients $c_{p,q} \in \mathbb{Z}$ such that

$$P_{c,T}^T(x_1, \dots, x_m) P_{c^\circ,T}^T(x_1, \dots, x_m) = \sum_{p,q=1}^m c_{p,q} x_p x_q \in \mathbb{Z}[x_1, \dots, x_m].$$

It follows from Lemma 6.2 in [20] that

$$\begin{aligned} \Lambda_{T'}(\mathbf{g}_{c,T}^{T'}, \mathbf{g}_{c^\circ,T}^{T'}) &= \Lambda_{T'} \left(P_{c,T}^T(\mathbf{g}_{1,T}^{T'}, \dots, \mathbf{g}_{m,T}^{T'}), P_{c^\circ,T}^T(\mathbf{g}_{1,T}^{T'}, \dots, \mathbf{g}_{m,T}^{T'}) \right) \\ &= \sum_{p,q=1}^m c_{p,q} \Lambda_{T'}(\mathbf{g}_{p,T}^{T'}, \mathbf{g}_{q,T}^{T'}) \\ &= \sum_{p,q=1}^m c_{p,q} \Lambda_T(\mathbf{e}_p, \mathbf{e}_q) \\ &= \Lambda_T(P_{c,T}^T(\mathbf{e}_1, \dots, \mathbf{e}_m), P_{c^\circ,T}^T(\mathbf{e}_1, \dots, \mathbf{e}_m)) \\ &= \Lambda_T(\mathbf{g}_{c,T}^T, \mathbf{g}_{c^\circ,T}^T). \end{aligned}$$

Let us write $\mathbf{g}_{\mathcal{C}}^T = \sum_{c \in \mathcal{C}} \mathbf{g}_{c,T}^T$ and $\mathbf{g}_{\mathcal{C}^\circ}^T = \sum_{c^\circ \in \mathcal{C}^\circ} \mathbf{g}_{c^\circ,T}^T$ and also $\mathbf{g}_{\mathcal{C}}^{T'} = \sum_{c \in \mathcal{C}} \mathbf{g}_{c,T}^{T'}$ and $\mathbf{g}_{\mathcal{C}^\circ}^{T'} = \sum_{c^\circ \in \mathcal{C}^\circ} \mathbf{g}_{c^\circ,T}^{T'}$. Then

$$\begin{aligned} \Lambda_{T'}(\mathbf{g}_{\mathcal{C}}^{T'}, \mathbf{g}_{\mathcal{C}^\circ}^{T'}) &= \sum_{c \in \mathcal{C}, c^\circ \in \mathcal{C}^\circ} \Lambda_{T'}(\mathbf{g}_{c,T}^{T'}, \mathbf{g}_{c^\circ,T}^{T'}) \\ &= \sum_{c \in \mathcal{C}, c^\circ \in \mathcal{C}^\circ} \Lambda_T(\mathbf{g}_{c,T}^T, \mathbf{g}_{c^\circ,T}^T) \\ &= \Lambda_T(\mathbf{g}_{\mathcal{C}}^T, \mathbf{g}_{\mathcal{C}^\circ}^T) \end{aligned}$$

as desired. □

We now come to the main definition of the present paper.

Definition 5.5. We will write $\mathbb{I}_{\mathcal{D}}^q$ for the element of $\mathcal{F}_{\mathcal{D}}$ given by

$$\mathbb{I}_{\mathcal{D}}^q(l) = \omega^{-N_l} \cdot [\mathcal{C}]^{-1} \otimes [\mathcal{C}^\circ].$$

Here we are regarding \mathcal{C} and \mathcal{C}° as simple multicurves on S and S° , respectively, and we are writing $[\mathcal{C}]$ and $[\mathcal{C}^\circ]$ for the corresponding classes in the skein algebra.

Our goal is to show that the above definition provides a map $\mathbb{I}_{\mathcal{D}}^q : \mathcal{D}_{PGL_2, S}(\mathbb{Z}^t) \rightarrow \mathcal{D}_{PGL_2, S}^q$. Let

$$\begin{aligned} \mathbf{w} &= (w_1, \dots, w_m), \\ \mathbf{w}^\circ &= (w_1^\circ, \dots, w_m^\circ) \end{aligned}$$

be integral vectors where w_i is the weight of the curve corresponding to the edge i of $T_{\mathcal{C}}$ and w_i° is the weight of the curve corresponding to the edge i of $T_{\mathcal{C}^\circ}$.

Theorem 5.6. Let T be an ideal triangulation of S , and let B_j, X_j ($j \in J$) be defined as above. Then for any $l \in \mathcal{D}_{PGL_2, S}(\mathbb{Z}^t)$, the element $\mathbb{I}_{\mathcal{D}}^q(l)$ is a rational expression in the B_i, X_i with coefficients in $\mathbb{Z}_{\geq 0}[q, q^{-1}]$.

Proof. By applying the relations from Definition 3.5, we can write

$$\mathbb{I}_{\mathcal{D}}^q(l) = \omega^{-N_l} \cdot \omega^\alpha \prod_{i=1}^m M_{T_{\mathcal{C}}}(\mathbf{e}_i)^{-w_i} \otimes \omega^{\alpha^\circ} \prod_{i=1}^m M_{T_{\mathcal{C}^\circ}}(\mathbf{e}_i)^{w_i^\circ}$$

where

$$\alpha = \sum_{i < j} \Lambda_{T_{\mathcal{C}}}(\mathbf{e}_i, \mathbf{e}_j) w_i w_j$$

and

$$\alpha^\circ = \sum_{i < j} \Lambda_{T_{\mathcal{C}^\circ}}(\mathbf{e}_i, \mathbf{e}_j) w_i^\circ w_j^\circ.$$

Then Corollary B.14 implies

$$\mathbb{I}_{\mathcal{D}}^q(l) = \omega^{-N_l + \alpha + \alpha^\circ} \cdot \prod_{i=1}^m (F_i \cdot M_T(\mathbf{g}_i))^{-w_i} \otimes \prod_{i=1}^m (F_i^\circ \cdot M_{T^\circ}(\mathbf{g}_i^\circ))^{w_i^\circ}.$$

Each F_i in the last line denotes a polynomial in the expressions $M_T(\sum_j \varepsilon_{kj} \mathbf{e}_j)$ with coefficients in $\mathbb{Z}_{\geq 0}[\omega^4, \omega^{-4}]$, and each F_i° denotes a polynomial in the expressions $M_{T^\circ}(\sum_j \varepsilon_{kj} \mathbf{e}_j)$ with coefficients in $\mathbb{Z}_{\geq 0}[\omega^4, \omega^{-4}]$. The \mathbf{g}_i and \mathbf{g}_i° are integral vectors. We have

$$\Lambda_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i, \mathbf{e}_j\right) = \sum_i \varepsilon_{ki} \lambda_{ij} = \sum_i b_{ik} \lambda_{ij} = 4\delta_{kj}$$

so that

$$M_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right) M_T(\mathbf{e}_j) = \omega^{-8\delta_{kj}} M_T(\mathbf{e}_j) M_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right).$$

Similarly, we have

$$M_{T^\circ}\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right) M_{T^\circ}(\mathbf{e}_j) = \omega^{8\delta_{kj}} M_{T^\circ}(\mathbf{e}_j) M_{T^\circ}\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right).$$

It follows that we can rearrange the factors in the last expression for $\mathbb{I}_{\mathcal{D}}^q(l)$ to get

$$\begin{aligned} \mathbb{I}_{\mathcal{D}}^q(l) &= \omega^{-N_l + \alpha + \alpha^\circ} P \cdot \prod_{i=1}^m M_T(\mathbf{g}_i)^{-w_i} \otimes Q \cdot \prod_{i=1}^m M_{T^\circ}(\mathbf{g}_i^\circ)^{w_i^\circ} \\ &= \omega^{-N_l + \alpha + \alpha^\circ} (P \otimes Q) \cdot \left(\prod_{i=1}^m M_T(\mathbf{g}_i)^{-w_i} \otimes \prod_{i=1}^m M_{T^\circ}(\mathbf{g}_i^\circ)^{w_i^\circ} \right) \end{aligned}$$

where P is a rational function in the expressions $M_T(\sum_i \varepsilon_{kj} \mathbf{e}_j)$ with coefficients in $\mathbb{Z}_{\geq 0}[\omega^4, \omega^{-4}]$ and Q is a polynomial in the expressions $M_{T^\circ}(\sum_i \varepsilon_{kj} \mathbf{e}_j)$ with coefficients in $\mathbb{Z}_{\geq 0}[\omega^4, \omega^{-4}]$. We claim that the factor $P \otimes Q$ is a rational function of the B_j and X_j . Indeed, $P \otimes 1$ is a rational function in the X_j , while $1 \otimes Q$ is a polynomial in the expressions

$$\begin{aligned} 1 \otimes M_{T^\circ}\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right) &= M_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i - \sum_i \varepsilon_{ki} \mathbf{e}_i\right) \otimes M_{T^\circ}\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right) \\ &= M_T\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right) M_T\left(-\sum_i \varepsilon_{ki} \mathbf{e}_i\right) \otimes M_{T^\circ}\left(\sum_i \varepsilon_{ki} \mathbf{e}_i\right) \\ &= X_k \mathbb{B}_k. \end{aligned}$$

Hence the product

$$P \otimes Q = (P \otimes 1) \cdot (1 \otimes Q)$$

is a rational expression as claimed. By Lemma 6.2 in [20], we have $\Lambda_{T_c}(\mathbf{e}_i, \mathbf{e}_j) = \Lambda_T(\mathbf{g}_i, \mathbf{g}_j)$, and so we can write

$$\mathbb{I}_{\mathcal{D}}(l) = \omega^{-N_l} (P \otimes Q) \cdot (M_T(-\mathbf{g}_c) \otimes M_{T^\circ}(\mathbf{g}_{c^\circ})).$$

We have

$$M_T(-\mathbf{g}_{c^\circ}) \otimes M_{T^\circ}(\mathbf{g}_{c^\circ}) = \prod_i B_i^{g_i}$$

for some integral exponents g_i , and so

$$\begin{aligned} \mathbb{I}_{\mathcal{D}}(l) &= \omega^{-N_l} (P \otimes Q) \cdot (M_T(-\mathbf{g}_c) M_T(\mathbf{g}_{c^\circ}) \otimes 1) \cdot \prod_i B_i^{g_i} \\ &= (P \otimes Q) \cdot (M_T(-\mathbf{g}_c + \mathbf{g}_{c^\circ}) \otimes 1) \cdot \prod_i B_i^{g_i}. \end{aligned}$$

It follows from Lemma 4.5 that the factor $M_T(-\mathbf{g}_c + \mathbf{g}_{c^\circ}) \otimes 1$ is a monomial in the X_j with coefficients in $\mathbb{Z}_{\geq 0}[\omega^4, \omega^{-4}]$. This completes the proof. \square

By the properties established in Propositions 5.2, 5.3, and 4.3, we can regard the rational expression of Theorem 5.6 as an element of the algebra $\mathcal{D}_{PGL_2,S}^q$. Thus we have constructed a canonical map $\mathbb{I}_{\mathcal{D}}^q : \mathcal{D}_{PGL_2,S}(\mathbb{Z}^t) \rightarrow \mathcal{D}_{PGL_2,S}^q$ as desired.

5.3 Properties

We will now examine the properties of our construction. To formulate our results precisely, we first note that there is a natural inclusion $\varphi : \mathcal{A}_{SL_2,S}^0(\mathbb{Z}^t) \hookrightarrow \mathcal{D}_{PGL_2,S}(\mathbb{Z}^t)$.

Definition 5.7. Let l be a point of $\mathcal{A}_{SL_2,S}^0(\mathbb{Z}^t)$ represented by a collection of curves on S . Draw those curves having positive weight on the disk S° . The point l determines a collection of edges of S having negative weight. Take absolute values to get a collection of positive weights for these edges of S . These data determine the point $\varphi(l) \in \mathcal{D}_{PGL_2,S}(\mathbb{Z}^t)$.

We also have a natural projection $\pi^q : \mathcal{D}_{PGL_2,S}^q \rightarrow \mathcal{X}_{SL_2,S}^q$.

Definition 5.8. Let m be a point of $\mathcal{D}_{PGL_2,S}^q$ represented as a rational expression in the generators B_i and X_i . Then $\pi^q(m)$ is obtained by setting all of the variables B_i in this rational expression equal to 1.

Theorem 5.9. The map $\mathbb{I}_{\mathcal{D}}^q : \mathcal{D}_{PGL_2,S}(\mathbb{Z}^t) \rightarrow \mathcal{D}_{PGL_2,S}^q$ satisfies the following properties:

1. Each $\mathbb{I}_{\mathcal{D}}^q(l)$ is a rational function in the variables B_i and X_i with coefficients in $\mathbb{Z}_{\geq 0}[q, q^{-1}]$.
2. The expression $\mathbb{I}_{\mathcal{D}}^q(l)$ agrees with the rational function $\mathbb{I}_D(l)$ when $q = 1$.
3. The following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_{SL_2,S}^0(\mathbb{Z}^t) & \xrightarrow{\mathbb{I}_{\mathcal{A}}^q} & \mathcal{X}_{PGL_2,S}^q \\ \varphi \downarrow & & \uparrow \pi^q \\ \mathcal{D}_{PGL_2,S}(\mathbb{Z}^t) & \xrightarrow{\mathbb{I}_{\mathcal{D}}^q} & \mathcal{D}_{PGL_2,S}^q \end{array}$$

Proof. 1. This was proved in Theorem 5.6.

2. This follows immediately from the definition of $\mathbb{I}_{\mathcal{D}}$ given in [1].

3. Let l be a point of $\mathcal{A}_{SL_2,S}^0(\mathbb{Z}^t)$, represented by a collection of curves on S . There exists an ideal triangulation T_l of \hat{S} such that each of these curves coincides with an edge of T_l . Let

$$\mathbf{w} = (w_1, \dots, w_m)$$

be the integral vector whose i th component w_i is the weight of the curve corresponding to the edge i of T_l . We will write $\mathbf{w}_+ = (w_1^+, \dots, w_m^+)$ for the vector whose i th component

equals w_i if $w_i \geq 0$ and zero otherwise. We will write $\mathbf{w}_- = (w_1^-, \dots, w_m^-)$ for the vector whose i th component equals w_i if $w_i \leq 0$ and zero otherwise. Then

$$\begin{aligned}
1 \otimes M_{T_l^\circ}(\mathbf{w}) &= 1 \otimes \omega^{\Lambda_{T_l^\circ}(\mathbf{w}_-, \mathbf{w}_+)} M_{T_l^\circ}(\mathbf{w}_-) M_{T_l^\circ}(\mathbf{w}_+) \\
&= 1 \otimes \omega^{\Lambda_{T_l^\circ}(\mathbf{w}_-, \mathbf{w}_+)} \omega^{\sum_{i < j} \Lambda_{T_l^\circ}(\mathbf{e}_i, \mathbf{e}_j) w_i^- w_j^-} \prod_{i=1}^m M_{T_l^\circ}(\mathbf{e}_i)^{w_i^-} \cdot M_{T_l^\circ}(\mathbf{w}_+) \\
&= \omega^{\Lambda_{T_l^\circ}(\mathbf{w}_-, \mathbf{w}_+)} \omega^{-\sum_{i < j} \Lambda_{T_l}(\mathbf{e}_i, \mathbf{e}_j) w_i^- w_j^-} \prod_{i=1}^m M_{T_l}(\mathbf{e}_{m-i+1})^{w_{m-i+1}^-} \otimes M_{T_l^\circ}(\mathbf{w}_+) \\
&= \omega^{\Lambda_{T_l^\circ}(\mathbf{w}_-, \mathbf{w}_+)} M_{T_l}(-\mathbf{w}_-)^{-1} \otimes M_{T_l^\circ}(\mathbf{w}_+)
\end{aligned}$$

in the algebra $\mathcal{F}_{\mathcal{D}}$ defined above. In the third step of this calculation, we have factored out $\prod_{i=1}^m (M_{T_l}(-\mathbf{e}_i) \otimes M_{T_l^\circ}(\mathbf{e}_i))^{w_i^-}$, which equals 1 in the quotient $\mathcal{F}_{\mathcal{D}} = (\mathcal{F} \otimes_{\mathbb{Z}[\omega, \omega^{-1}]} \mathcal{F}^\circ) / \mathcal{I}$. It is easy to see that the expression in the last line of this calculation is $(\mathbb{I}_{\mathcal{D}}^q \circ \varphi)(l)$. That is,

$$(\mathbb{I}_{\mathcal{D}}^q \circ \varphi)(l) = 1 \otimes M_{T_l^\circ}(\mathbf{w}).$$

By Theorem 4.6, we know that $1 \otimes M_{T_l^\circ}(\mathbf{w})$ is a Laurent polynomial in the expressions $1 \otimes M_{T_l^\circ}(\sum_i \varepsilon_{ki} \mathbf{e}_i) = X_k \mathbb{B}_k$. If we set all of the variables B_i equal to 1, we recover the Laurent polynomial $\mathbb{I}_{\mathcal{A}}^q(l)$. Thus we have

$$(\pi^q \circ \mathbb{I}_{\mathcal{D}}^q \circ \varphi)(l) = \mathbb{I}_{\mathcal{A}}^q(l)$$

as desired. \square

A Derivation of mutation formulas

In this appendix, we calculate the action of the map μ_k^q of Definition 2.7 on the generators B_i and X_i .

Lemma A.1. Let $\varphi(x)$ be any formal power series in x . Then

$$\varphi(X_k) B_i = B_i \varphi(q^{2\delta_{ik}} X_k), \quad \varphi(\widehat{X}_k) B_i = B_i \varphi(q^{2\delta_{ik}} \widehat{X}_k),$$

and

$$\varphi(X_k) X_i = X_i \varphi(q^{2\varepsilon_{ki}} X_k).$$

Proof. Write $\varphi(x) = a_0 + a_1 x + a_2 x^2 + \dots$. Then the relation $X_k X_i = q^{2\varepsilon_{ki}} X_i X_k$ implies

$$\begin{aligned}
\varphi(X_k) X_i &= a_0 X_i + a_1 X_k X_i + a_2 X_k^2 X_i + \dots \\
&= X_i a_0 + X_i a_1 q^{2\varepsilon_{ki}} X_k + X_i a_2 q^{4\varepsilon_{ki}} X_k^2 + \dots \\
&= X_i (a_0 + a_1 (q^{2\varepsilon_{ki}} X_k) + a_2 (q^{2\varepsilon_{ki}} X_k)^2 + \dots) \\
&= X_i \varphi(q^{2\varepsilon_{ki}} X_k).
\end{aligned}$$

This proves the third relation. The first two relations are proved similarly. For these one uses the facts $X_k B_i = q^{2\delta_{ik}} B_i X_k$ and $\widehat{X}_k B_i = q^{2\delta_{ik}} B_i \widehat{X}_k$. \square

Proposition A.2. The map μ_k^\sharp is given on generators by the formulas

$$\mu_k^\sharp(B_i) = \begin{cases} B_k(1 + qX_k)(1 + q\widehat{X}_k)^{-1} & \text{if } i = k \\ B_i & \text{if } i \neq k \end{cases}$$

and

$$\mu_k^\sharp(X_i) = \begin{cases} X_i(1 + qX_k)(1 + q^3X_k) \dots (1 + q^{2|\varepsilon_{ik}|-1}X_k) & \text{if } \varepsilon_{ik} \leq 0 \\ X_i((1 + q^{-1}X_k)(1 + q^{-3}X_k) \dots (1 + q^{1-2|\varepsilon_{ik}|}X_k))^{-1} & \text{if } \varepsilon_{ik} \geq 0. \end{cases}$$

Proof. By Lemma A.1, we have

$$\begin{aligned} \mu_k^\sharp(B_i) &= \Psi^q(X_k)\Psi^q(\widehat{X}_k)^{-1}B_i\Psi^q(\widehat{X}_k)\Psi^q(X_k)^{-1} \\ &= B_i\Psi^q(q^{2\delta_{ik}}X_k)\Psi^q(q^{2\delta_{ik}}\widehat{X}_k)^{-1}\Psi^q(\widehat{X}_k)\Psi^q(X_k)^{-1}. \end{aligned}$$

If $i \neq k$, then this equals B_i as desired. Suppose on the other hand that $i = k$. Using the identity $\Psi^q(q^2x) = (1 + qx)\Psi^q(x)$ and commutativity of X_i and \widehat{X}_k , we can rewrite this last expression as

$$\begin{aligned} \mu_k^\sharp(B_i) &= B_k\Psi^q(q^2X_k)\Psi^q(q^2\widehat{X}_k)^{-1}\Psi^q(\widehat{X}_k)\Psi^q(X_k)^{-1} \\ &= B_k(1 + qX_k)\Psi^q(X_k) \left((1 + q\widehat{X}_k)\Psi^q(\widehat{X}_k) \right)^{-1} \Psi^q(\widehat{X}_k)\Psi^q(X_k)^{-1} \\ &= B_k(1 + qX_k)(1 + q\widehat{X}_k)^{-1}. \end{aligned}$$

This completes the proof of the first formula.

To prove the second formula, observe that by Lemma A.1 and the commutativity of X_i and \widehat{X}_k we have

$$\begin{aligned} \mu_k^\sharp(X_i) &= \Psi^q(X_k)\Psi^q(\widehat{X}_k)^{-1}X_i\Psi^q(\widehat{X}_k)\Psi^q(X_k)^{-1} \\ &= \Psi^q(X_k)X_i\Psi^q(X_k)^{-1} \\ &= X_i\Psi^q(q^{-2\varepsilon_{ik}}X_k)\Psi^q(X_k)^{-1}. \end{aligned}$$

If $\varepsilon_{ik} \leq 0$, then the identity $\Psi^q(q^2x) = (1 + qx)\Psi^q(x)$ implies

$$\begin{aligned} \mu_k^\sharp(X_i) &= X_i\Psi^q(q^2q^{2|\varepsilon_{ik}|-2}X_k)\Psi^q(X_k)^{-1} \\ &= X_i(1 + q^{2|\varepsilon_{ik}|-1}X_k)\Psi^q(q^{2|\varepsilon_{ik}|-2}X_k)\Psi^q(X_k)^{-1} \\ &= X_i(1 + q^{2|\varepsilon_{ik}|-1}X_k)(1 + q^{2|\varepsilon_{ik}|-3}X_k)\Psi^q(q^{2|\varepsilon_{ik}|-4}X_k)\Psi^q(X_k)^{-1} \\ &= \dots \\ &= X_i(1 + q^{2|\varepsilon_{ik}|-1}X_k) \dots (1 + q^3X_k)(1 + qX_k) \end{aligned}$$

as desired. If $\varepsilon_{ik} \geq 0$, there is a similar argument using the identity $\Psi^q(q^{-2}x) = (1 + q^{-1}x)^{-1}\Psi^q(x)$. \square

Lemma A.3. Let $(\Lambda, \{e_i\}_{i \in I}, \{e_j\}_{j \in J}, (\cdot, \cdot))$ be a seed. If we mutate this seed in the direction of a basis vector e_k , then the basis $\{f_i\}$ for Λ^\vee transforms to a new basis $\{f'_i\}$ given by

$$f' = \begin{cases} -f_i + \sum_j [-\varepsilon_{kj}]_+ f_j & \text{if } i = k \\ f_i & \text{if } i \neq k. \end{cases}$$

Proof. The transformation $e_i \mapsto e'_i$ can be represented by an explicit matrix by Definition 2.2, and the transformation rule appearing in the lemma is represented by the transpose of this matrix. \square

Proposition A.4. The map μ'_k is given on generators by the formulas

$$\mu'_k(B'_i) = \begin{cases} \mathbb{B}_k^- / B_k & \text{if } i = k \\ B_i & \text{if } i \neq k \end{cases}$$

and

$$\mu'_k(X'_i) = \begin{cases} X_k^{-1} & \text{if } i = k \\ q^{-[\varepsilon_{ik}] + \varepsilon_{ik}} X_i X_k^{[\varepsilon_{ik}] +} & \text{if } i \neq k. \end{cases}$$

Proof. Let Y_v be the generator of \mathcal{D}_i associated to $v \in \Lambda_{\mathcal{D}}$ as in Definition 2.5. By Lemma A.3 and the fact that $(f_i, f_j)_{\mathcal{D}} = 0$ for all i, j , we have

$$\begin{aligned} \mu'_k(B'_k) &= Y_{-f_k + \sum_j [-\varepsilon_{kj}]_+ f_j} \\ &= B_k^{-1} \mathbb{B}_k^-. \end{aligned}$$

Similarly we have

$$\begin{aligned} \mu'_k(X'_i) &= Y_{e_i + [\varepsilon_{ik}]_+ e_k} \\ &= q^{-[\varepsilon_{ik}] + (e_i, e_k)_{\mathcal{D}}} Y_{e_i} Y_{[\varepsilon_{ik}]_+ e_k} \\ &= q^{-[\varepsilon_{ik}] + \varepsilon_{ik}} X_i X_k^{[\varepsilon_{ik}] +} \end{aligned}$$

for $i \neq k$ and $\mu'_k(X'_i) = Y_{-e_k} = X_k^{-1}$ for $i = k$. \square

Theorem A.5. The map μ_k^q is given on generators by the formulas

$$\mu_k^q(B'_i) = \begin{cases} (qX_k \mathbb{B}_k^+ + \mathbb{B}_k^-) B_k^{-1} (1 + q^{-1} X_k)^{-1} & \text{if } i = k \\ B_i & \text{if } i \neq k \end{cases}$$

and

$$\mu_k^q(X'_i) = \begin{cases} X_i \prod_{p=0}^{|\varepsilon_{ik}|-1} (1 + q^{2p+1} X_k) & \text{if } \varepsilon_{ik} \leq 0 \text{ and } i \neq k \\ X_i X_k^{\varepsilon_{ik}} \prod_{p=0}^{\varepsilon_{ik}-1} (X_k + q^{2p+1})^{-1} & \text{if } \varepsilon_{ik} \geq 0 \text{ and } i \neq k \\ X_k^{-1} & \text{if } i = k. \end{cases}$$

Proof. By our formulas for μ'_k and μ_k^\sharp , we have

$$\begin{aligned}\mu_k^q(B'_k) &= \mu_k^\sharp(\mu'_k(B'_k)) = \mu_k^\sharp(\mathbb{B}_k^-/B_k) \\ &= \mathbb{B}_k^- \left(B_k(1 + qX_k)(1 + q\widehat{X}_k)^{-1} \right)^{-1} \\ &= \mathbb{B}_k^- (1 + q\widehat{X}_k)(1 + qX_k)^{-1} B_k^{-1}.\end{aligned}$$

By the definitions of \mathbb{B}_k^- and \widehat{X}_k , this equals

$$\begin{aligned}\mu_k^q(B'_k) &= \prod_{i|\varepsilon_{ki}<0} B_i^{-\varepsilon_{ki}} (1 + qX_k \prod_i B_i^{\varepsilon_{ki}}) (1 + qX_k)^{-1} B_k^{-1} \\ &= \left(\prod_{i|\varepsilon_{ki}<0} B_i^{-\varepsilon_{ki}} + qX_k \prod_{i|\varepsilon_{ki}>0} B_i^{\varepsilon_{ki}} \right) (1 + qX_k)^{-1} B_k^{-1} \\ &= (qX_k \mathbb{B}_k^+ + \mathbb{B}_k^-) (1 + qX_k)^{-1} B_k^{-1} \\ &= (qX_k \mathbb{B}_k^+ + \mathbb{B}_k^-) B_k^{-1} (1 + q^{-1}X_k)^{-1}.\end{aligned}$$

From this calculation, we easily obtain the formula describing the action of the map μ_k^q on the generators B'_i .

On the other hand, if $\varepsilon_{ik} \leq 0$ and $i \neq k$, then

$$\begin{aligned}\mu_k^q(X'_i) &= \mu_k^\sharp(\mu'_k(X'_i)) = \mu_k^\sharp(X_i) \\ &= X_i(1 + qX_k)(1 + q^3X_k) \dots (1 + q^{2|\varepsilon_{ik}|-1}X_k) \\ &= X_i \prod_{p=0}^{|\varepsilon_{ik}|-1} (1 + q^{2p+1}X_k).\end{aligned}$$

If $\varepsilon_{ik} \geq 0$ and $i \neq k$, then

$$\begin{aligned}\mu_k^q(X'_i) &= \mu_k^\sharp(\mu'_k(X'_i)) = \mu_k^\sharp(q^{-\varepsilon_{ik}^2} X_i X_k^{\varepsilon_{ik}}) \\ &= q^{-\varepsilon_{ik}^2} X_i \left((1 + q^{-1}X_k)(1 + q^{-3}X_k) \dots (1 + q^{1-2|\varepsilon_{ik}|}X_k) \right)^{-1} X_k^{\varepsilon_{ik}} \\ &= q^{-\varepsilon_{ik}^2} X_i \left(q^{-1}(X_k + q)q^{-3}(X_k + q^3) \dots q^{1-2|\varepsilon_{ik}|}(X_k + q^{2|\varepsilon_{ik}|-1}) \right)^{-1} X_k^{\varepsilon_{ik}}.\end{aligned}$$

Applying the identity $1 + 3 + 5 + \dots + (2N - 1) = N^2$, this becomes

$$\begin{aligned}\mu_k^q(X'_i) &= X_i \left((X_k + q)(X_k + q^3) \dots (X_k + q^{2|\varepsilon_{ik}|-1}) \right)^{-1} X_k^{\varepsilon_{ik}} \\ &= X_i X_k^{\varepsilon_{ik}} (X_k + q)^{-1} (X_k + q^3)^{-1} \dots (X_k + q^{2|\varepsilon_{ik}|-1})^{-1} \\ &= X_i X_k^{\varepsilon_{ik}} \prod_{p=0}^{\varepsilon_{ik}-1} (X_k + q^{2p+1})^{-1}.\end{aligned}$$

Finally, if $i = k$, then we have

$$\mu_k^q(X'_i) = \mu_k^\sharp(\mu'_k(X'_i)) = \mu_k^\sharp(X_k^{-1}) = X_k^{-1}.$$

This proves the second formula. □

B Review of quantum cluster algebras

B.1 General theory of quantum cluster algebras

Here we review the theory of quantum cluster algebras, following [4, 20]. Throughout this section, m and n will be positive integers with $m \geq n$.

Definition B.1. Let $k \in \{1, \dots, n\}$. We say that an $m \times n$ matrix $\mathbf{B}' = (b'_{ij})$ is obtained from an $m \times n$ matrix $\mathbf{B} = (b_{ij})$ by *matrix mutation* in the direction k if the entries of \mathbf{B}' are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{if } k \notin \{i, j\}. \end{cases}$$

In this case, we write $\mu_k(\mathbf{B}) = \mathbf{B}'$.

Definition B.2. Let $\mathbf{B} = (b_{ij})$ be an $m \times n$ integer matrix, and let $\Lambda = (\lambda_{ij})$ be a skew-symmetric $m \times m$ integer matrix. We say that the pair (Λ, \mathbf{B}) is *compatible* if for each $j \in \{1, \dots, n\}$ and $i \in \{1, \dots, m\}$, we have

$$\sum_{k=1}^m b_{kj} \lambda_{ki} = \delta_{ij} d_j$$

for some positive integers d_j ($j \in \{1, \dots, n\}$). Equivalently, the product $\mathbf{B}^t \Lambda$ equals the $n \times m$ matrix $(D|0)$ where D is the $n \times n$ diagonal matrix with diagonal entries d_1, \dots, d_n .

Let $k \in \{1, \dots, n\}$ and choose a sign $\epsilon \in \{\pm 1\}$. Denote by E_ϵ the $m \times m$ matrix with entries given by

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k \\ -1 & \text{if } i = j = k \\ \max(0, -\epsilon b_{ik}) & \text{if } i \neq j = k \end{cases}$$

and set

$$\Lambda' = E_\epsilon^t \Lambda E_\epsilon.$$

Proposition B.3 ([4], Proposition 3.4). The matrix Λ' is skew-symmetric and independent of the sign ϵ . Moreover, $(\Lambda', \mu_k(\mathbf{B}))$ is a compatible pair.

Definition B.4 ([4], Definition 3.5). Let (Λ, \mathbf{B}) be a compatible pair and let $k \in \{1, \dots, n\}$. We say that the pair $(\Lambda', \mu_k(\mathbf{B}))$ is obtained from (Λ, \mathbf{B}) by *mutation* in the direction k and write $\mu_k(\Lambda, \mathbf{B}) = (\Lambda', \mu_k(\mathbf{B}))$.

Let L be a lattice of rank m equipped with a skew-symmetric bilinear form $\Lambda : L \times L \rightarrow \mathbb{Z}$. Let ω be a formal variable. We can associate to these data a quantum torus algebra \mathcal{T} . It is generated over $\mathbb{Z}[\omega, \omega^{-1}]$ by variables A^v ($v \in \Lambda$) subject to the commutation relations

$$A^{v_1} A^{v_2} = \omega^{-\Lambda(v_1, v_2)} A^{v_1 + v_2}.$$

In the literature on quantum cluster algebras, this quantum torus algebra is typically called a *based quantum torus*, and the parameter is denoted $q^{-1/2}$, rather than ω . (See [4, 17, 20] for example.) This quantum torus algebra has a noncommutative fraction field which we denote \mathcal{F} .

Definition B.5. A *toric frame* in \mathcal{F} is a mapping $M : \mathbb{Z}^m \rightarrow \mathcal{F} - \{0\}$ of the form

$$M(v) = \phi(A^{\eta(v)})$$

where ϕ is an automorphism of \mathcal{F} and $\eta : \mathbb{Z}^m \rightarrow L$ is an isomorphism of lattices.

Note that the image $M(\mathbb{Z}^m)$ of a toric frame is a basis for an isomorphic copy of \mathcal{T} in \mathcal{F} . We have the relations

$$\begin{aligned} M(v_1)M(v_2) &= \omega^{-\Lambda_M(v_1, v_2)} M(v_1 + v_2), \\ M(v_1)M(v_2) &= \omega^{-2\Lambda_M(v_1, v_2)} M(v_2)M(v_1), \\ M(v)^{-1} &= M(-v), \\ M(0) &= 1 \end{aligned}$$

where the form Λ_M on \mathbb{Z}^m is obtained from Λ using the isomorphism η .

Definition B.6. A *quantum seed* is a pair (M, \mathbf{B}) where M is a toric frame in \mathcal{F} and \mathbf{B} is an $m \times n$ integer matrix such that (Λ_M, \mathbf{B}) is a compatible pair.

Definition B.7. Let (M, \mathbf{B}) be a quantum seed and write $\mathbf{B} = (b_{ij})$. For any index $k \in \{1, \dots, k\}$ and any sign $\epsilon \in \{\pm 1\}$, we define a mapping $M' : \mathbb{Z}^m \rightarrow \mathcal{F} - \{0\}$ by the formulas

$$\begin{aligned} M'(v) &= \sum_{p=0}^{v_k} \binom{v_k}{p}_{\omega^{-d_k}} M(E_\epsilon v + \epsilon p b^k), \\ M'(-v) &= M'(v)^{-1} \end{aligned}$$

where $v = (v_1, \dots, v_m) \in \mathbb{Z}^m$ is such that $v_k \geq 0$ and b^k denotes the k th column of \mathbf{B} . Here the t -binomial coefficient is given by

$$\binom{r}{p}_t = \frac{(t^r - t^{-r}) \dots (t^{r-p+1} - t^{-r+p-1})}{(t^p - t^{-p}) \dots (t - t^{-1})}.$$

Proposition B.8 ([4], Proposition 4.7). The mapping M' satisfies the following properties:

1. The mapping M' is a toric frame which is independent of the sign ϵ .
2. The pair $(\Lambda_{M'}, \mu_k(\mathbf{B}))$ is compatible and obtained from the pair (Λ_M, \mathbf{B}) by mutation in the direction k .
3. The pair $(M', \mu_k(\mathbf{B}))$ is a quantum seed.

Definition B.9. Let (M, \mathbf{B}) be a quantum seed, and let $k \in \{1, \dots, n\}$. Let M' be the mapping from Definition B.7, and let $\mathbf{B}' = \mu_k(\mathbf{B})$. Then we say that the quantum seed (M', \mathbf{B}') is obtained from (M, \mathbf{B}) by *mutation* in the direction k .

Proposition B.10 ([4], Proposition 4.9). Let (M, \mathbf{B}) be a quantum seed, and suppose that (M', \mathbf{B}') is obtained from (M, \mathbf{B}) by mutation in the direction k . Then

$$M'(\mathbf{e}_k) = M\left(-\mathbf{e}_k + \sum_{i=1}^m [b_{ik}]_+ \mathbf{e}_i\right) + M\left(-\mathbf{e}_k + \sum_{i=1}^m [-b_{ik}]_+ \mathbf{e}_i\right)$$

and $M'(\mathbf{e}_i) = M(\mathbf{e}_i)$ for $i \neq k$.

Definition B.11. Denote by \mathbb{T}_n an n -regular tree with edges labeled by the numbers $1, \dots, n$ in such a way that the n edges emanating from any vertex have distinct labels. A *quantum cluster pattern* is an assignment of a quantum seed $\Sigma_t = (M_t, \mathbf{B}_t)$ to each vertex $t \in \mathbb{T}_n$ so that if t and t' are vertices connected by an edge labeled k , then $\Sigma_{t'}$ is obtained from Σ_t by a mutation in the direction k .

Given a quantum cluster pattern, let us define $A_{j;t} = M_t(\mathbf{e}_j)$. For $j \in \{n+1, \dots, m\}$, we have $A_{j;t} = A_{j;t'}$ for all $t, t' \in \mathbb{T}_n$, so we may omit one of the subscripts and write $A_j = A_{j;t}$ for all $t \in \mathbb{T}_n$. Let

$$\mathcal{S} = \{A_{j;t} : j \in \{1, \dots, n\}, t \in \mathbb{T}_n\}.$$

Definition B.12 ([4], Definition 4.12). Given a quantum cluster pattern $t \mapsto (M_t, \mathbf{B}_t)$, the associated *quantum cluster algebra* \mathcal{A} is the $\mathbb{Z}[\omega^{\pm 1}, A_{n+1}^{\pm 1}, \dots, A_m^{\pm 1}]$ -subalgebra of the ambient skew-field \mathcal{F} generated by elements of \mathcal{S} .

B.2 Quantum F -polynomials

One of the important tools that we apply in our construction of the map $\mathbb{I}_{\mathcal{D}}^q$ is the notion of a quantum F -polynomial from [20]. This extends Fomin and Zelevinsky's notion of F -polynomial [12] to the noncommutative setting and allows us to express a generator $A_{j;t}$ of a quantum cluster algebra in terms of the generators associated with an initial seed.

Theorem B.13 ([20], Theorem 5.3). Let (M_0, \mathbf{B}_0) be an initial quantum seed in a quantum cluster algebra \mathcal{A} and write $\mathbf{B}_0 = (b_{ij})$. Then there exists an integer $\lambda_{j;t} \in \mathbb{Z}$ and a polynomial $F_{j;t}$ in the variables

$$Y_j = M_0\left(\sum_i b_{ij} \mathbf{e}_i\right)$$

with coefficients in $\mathbb{Z}[\omega, \omega^{-1}]$ such that the cluster variable $A_{j;t} \in \mathcal{A}$ is given by

$$A_{j;t} = \omega^{\lambda_{j;t}} F_{j;t} \cdot M_0(\mathbf{g}_{j;t})$$

where $\mathbf{g}_{j;t}$ denotes the extended \mathbf{g} -vector of $A_{j;t}$.

The polynomial $F_{j;t}$ appearing in the theorem is known as a *quantum F -polynomial*. For the quantum cluster algebras considered in this paper, we have the following refinement of Theorem B.13.

Corollary B.14. Let \mathcal{A} be a quantum cluster algebra of type A_n , and suppose the matrix D appearing in the compatibility condition of Definition B.2 is four times the identity. Then there exists a polynomial $F_{j;t}$ in the variables Y_1, \dots, Y_n with coefficients in $\mathbb{Z}_{\geq 0}[\omega^4, \omega^{-4}]$ such that the cluster variable $A_{j;t} \in \mathcal{A}$ is given by

$$A_{j;t} = F_{j;t} \cdot M_0(\mathbf{g}_{j;t})$$

where $\mathbf{g}_{j;t}$ denotes the extended \mathbf{g} -vector of $A_{j;t}$.

Proof. For algebras of type A_n , it is known that each classical F -polynomial has nonzero constant term (for example by [18]). Hence, by [20], Theorem 6.1, we have $\lambda_{j;t} = 0$ in Theorem B.13. By [20], Theorem 7.4, we know that the coefficients of $F_{j;t}$ are Laurent polynomials in ω^4 with positive integral coefficients. \square

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References

- [1] Allegretti, D.G.L. (2014). Laminations from the symplectic double. [arXiv:1410.3035 \[math.GT\]](#).
- [2] Allegretti, D.G.L. (2015). The cluster symplectic double and moduli spaces of local systems. [arXiv:1509.01342 \[math.AG\]](#).
- [3] Allegretti, D.G.L. and Kim, H.K. (2015). A duality map for quantum cluster varieties from surfaces. [arXiv:1509.01567 \[math.QA\]](#).
- [4] Berenstein, A. and Zelevinsky, A. (2005). Quantum cluster algebras. *Advances in Mathematics*, **195**(2), 405–455.
- [5] Bonahon, F. and Wong, H. (2011). Quantum traces for representations of surface groups in $SL_2(\mathbb{C})$. *Geometry and Topology*, **15**(3), 1569–1615.
- [6] Fock, V.V. and Goncharov, A.B. (2006). Moduli spaces of local systems and higher Teichmüller theory. *Publications Mathématiques de l’Institut des Hautes Études Scientifiques*, **103**(1), 1–211.

- [7] Fock, V.V. and Goncharov, A.B. (2007). Dual Teichmüller and lamination spaces. In Handbook of Teichmüller theory I, *IRMA Lectures in Mathematics and Theoretical Physics*, **11**, 647–684.
- [8] Fock, V.V. and Goncharov, A.B. (2009). Cluster ensembles, quantization and the dilogarithm. *Annales Scientifiques de l'École Normale Supérieure*, **42**(6), 865–930.
- [9] Fock, V.V. and Goncharov, A.B. (2009). The quantum dilogarithm and representations of quantum cluster varieties. *Inventiones mathematicae*, **175**(2), 223–286.
- [10] Fock, V.V. and Goncharov, A.B. Symplectic double for moduli spaces of G -local systems on surfaces. [arXiv:1410.3526](#) [math.AG].
- [11] Fomin, S. and Zelevinsky, A. (2002). Cluster algebras I: Foundations. *Journal of the American Mathematical Society*, **15**(2), 497–529.
- [12] Fomin, S. and Zelevinsky, A. (2007). Cluster algebras IV: Coefficients. *Compositio Mathematica*, **143**(01), 112–164.
- [13] Gaiotto, D., Moore, G.W., and Neitzke, A. (2013). Framed BPS states. *Advances in Theoretical and Mathematical Physics*, **17**(2), 241–397.
- [14] Gross, M., Hacking, P., Keel, S., and Kontsevich, M. (2014). Canonical bases for cluster algebras. [arXiv:1411.1394](#) [math.AG].
- [15] Lê, T. (2015). Quantum Teichmüller spaces and quantum trace map. [arXiv:1511.06054](#) [math.GT].
- [16] Lusztig, G. (1990). Canonical bases arising from quantized enveloping algebras. *Journal of the American Mathematical Society*, **3**(2), 447–498.
- [17] Muller, G. (2012). Skein algebras and cluster algebras of marked surfaces. [arXiv:1204.0020](#) [math.QA].
- [18] Musiker, G. Schiffler, R., and Williams, L. (2011). Positivity for cluster algebras from surfaces. *Advances in Mathematics*, **227**(6), 2241–2308.
- [19] Thurston, D. (2014). Positive basis for surface skein algebras. *Proceedings of the National Academy of Sciences*, **111**(27), 9725–9732.
- [20] Tran, T. (2011). F -polynomials in quantum cluster algebras. *Algebras and representation theory*, **14**(6), 1025–1061.